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TABLE OF CONTENTS

Page

Mal'cev, A. I. On some classes of infinite soluble groups. [Мальцев, А. И. О некоторых классах бесконечных разрешимых групп.] Mat. Sbornik N.S. 28(70), 567–588 (1951). 1

Kulikov, L. Ya. On direct decompositions of groups. [Куликов, Л. Я. О прямых разложениях групп.] Ukrain. Mat. Ž. 4, 230–275, 347–372 (1952). 23

Golovin, O. N. Nilpotent products of groups. [Головин, О. Н. Нильпотентные произведения групп.] Mat. Sbornik N.S. 27(69), 427–454 (1950). 89

Golovin, O. N. The metabelian products of groups. [Головин, О. Н. Метабелевы произведения групп.] Mat. Sbornik N.S. 28(70), 431–444 (1951). 117

Golovin, O. N. On the isomorphism of nilpotent decompositions of groups. [Головин, О. Н. К вопросу об изоморфизме нильпотентных разложений группы.] Mat. Sbornik N.S. 28(70), 445–452 (1951). 133

Naĭmark, M. A. Description of all irreducible unitary representations of the classical groups. [Наймарк, М. А. Описание всех неприводимых унитарных предсеавлений классических групп.] Doklady Akad. Nauk SSSR. (N.S.) 84, 883–886 (1952). 141

Gel'fand, I. M., and Graev, M. I. Unitary representations of the real unimodular group (principal nondegenerate series). [Гельфанд, И. М. и Граев, М. И. Унитарные представления вещественной унимодулярной группы (основные невырожденные серии).] Izv. Akad. Nauk SSSR. Ser. Mat. 17, 189–248 (1953). 147

Gel'fand, I. M., and Šapiro, Z. Ya. Representations of the group of rotations in three-dimensional space and their applications. [Гельфанд, И. М. и Шапиро, З. Я. Представления группы вращений трёхмерного пространства и их применения.] Uspehi Mat. Nauk (N.S.) 7, no. 1(47), 3–117 (1952). 207

ON CERTAIN CLASSES OF INFINITE SOLVABLE GROUPS

A. I. MAL'CEV

The theory of solvable and nilpotent groups forms an important chapter in the theory of finite groups. Carrying over the ideas of solvability and nilpotency to infinite groups can take place by using one or another of the properties in the case of finite groups.

In the present paper we lay down the following definitions.

The finite or infinite group G is called *solvable* if it has a finite normal chain of subgroups $G \supset G_1 \cdots \supset G_n \supset 1$ with abelian factors. The group G will be called *nilpotent* if it contains subgroups G_t forming a finite central series; i.e., G_t/G_{t+1} lies in the center of $G/G_{t+1}(t = 0, 1, \cdots, n)$; $G_0 = G$; $G_{n+1} = 1$.

Solvable groups which possess a finite normal series with cyclic factors were studied under the name *S-groups* by Hirsch [3] (see also Zappa [12]). Various generalizations of the concept of solvable and nilpotent groups are discussed in the survey of A. G. Kuroš and S. N. Černikov [5], which contains further references. In particular, the group G is called *locally solvable* if every finite set of its elements generates a solvable subgroup. In the works of O. Yu Schmidt ([15], [16]), S. N. Černikov ([13], [14]), I. D. Ado ([1]) and others, there are given a series of conditions which locally solvable or locally nilpotent groups must satisfy to be solvable. Ordinarily, the groups considered by these authors are periodic. We can free ourselves of that restriction by making use of the concept of rank [7]. In the simplest case of locally nilpotent groups, that was done by N. N. Myagkova [11].

In the present work we consider the wider class of solvable groups, and we look for conditions under which locally solvable groups are solvable and other problems or properties concerning these same groups. Basic results formulated in [9] and Theorem 6 of [9] are presented in stronger form in the present paper in Theorem 7 and its corollary. The proof makes significant use of properties of groups of matrices; and certain results appear as corresponding theorems on matrix groups.

§1. Linear solvable groups

The set of all triangular matrices of a fixed order n with elements from an arbitrary field is a solvable group, the n-th commutator of which is equal to 1*. The first commutator of this group consists of the triangular matrices of special type—those with 1 on the main diagonal. If one forms the lower central series of commutators, then its n-th member will be 1, since every group of special triangular matrices is nilpotent. The following theorem states that, in a certain sense, the triangular form is the normal form for solvable groups of matrices.

Theorem 1. *Every solvable group of matrices with elements from an algebraically closed field contains a subgroup of finite index the matrices of which can simultaneously be reduced to triangular form.*

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* By definition a matrix is triangular if all elements below the main diagonal are equal to 0.

To prove this theorem, we employ

Lemma 1. *Let G be a periodic group of matrices of degree n; let every matrix have order dividing the fixed number N, which is relatively prime to the characteristic of the field, if the latter is finite. Then the group G is finite and its order does not exceed a certain bound depending only on n and N and independent of the characteristic of the field.*

Let H be an arbitrary subgroup of G, generated by a finite number of elements. The order of H is finite (see [17] and [6]) and relatively prime to the characteristic. Thus every irreducible representation of H in the given field can be obtained by the reduction of one of the non-modular irreducible representations of H (see [2], p. 74). Thus, the group H is isomorphic to a non-modular representation of degree N. By Jordan's theorem, H has an abelian normal subgroup A, of index not exceeding the number k_n, which depends only on n. The elements of A satisfy the relation $x^N = 1$, and their number thus cannot exceed N^n. Thus, the order of an arbitrary finite subgroup of the group G cannot exceed $k_n N^n$, and thus, the number of elements of G cannot exceed $k_n N^n$.

Proof of Theorem 1. The theorem is an obvious consequence of the following assertion: every absolutely irreducible solvable group of matrices contains a reducible subgroup of finite index. This assertion will now be proved. In the proof it is necessary to rely on the theorem of Clifford, which states that if a group of matrices is completely reducible, then every normal divisor is also completely reducible.

Let G be an irreducible solvable group of matrices over an algebraically closed field. First, suppose that G has an abelian normal divisor A not lying in the center. By Jordan's theorem, the matrices of the group A can be simultaneously transformed into diagonal form. Let $a = a_1 e_{11} + \cdots + a_n e_{nn}$ be an element of A not lying in the center of G (e_{ij} being the matrix units), so that surely for some i, j, a_i is not equal to a_j. Since A is a normal divisor of G, then for any g in G, the relation $g^{-1}ag = a_{i_1} e_{11} + \cdots + a_{i_n} e_{nn}$ must hold.* The mapping

$$g \rightarrow \begin{pmatrix} 1 & 2 & . & . & . & n \\ i_1 & i_2 & . & . & . & i_n \end{pmatrix}$$

is a many to one homomorphism from the group G to the symmetric group of degree n. Let R be the set of all elements of G which are mapped into the identity substitution. Since the symmetric group is finite, R has finite index in G. On the other hand, the elements of R are permutable with the not scalar matrix A, and hence, the group R is reducible.

Next, let us consider the case when all abelian normal divisors of the group G are contained in its center Z. Let H be the set of matrices of the form $a^{-1}g$, where a is the n-th root of the determinant of the element g of G. It is clear that with G also H is an irreducible solvable group. Let g_1, g_2 be two elements of G; and let the n-th roots of their determinants be a_1 and a_2 respectively. If the relation $a_1^{-1}g_1 = a_2^{-1}g_2$ holds, then g_1 and g_2 represent one and the same coset with respect to Z. Also, if the relation $g_2 = g_1 z$ holds, with z in Z, then for suitable

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* Translator's note: The argument seems incomplete.

a_1, a_2, the elements $a_1^{-1}g_1$ and $a_2^{-1}g_2$ will be equal, and a_i will be an n-th root of the determinant of $g_i (i = 1, 2)$. Thus, the correspondence $a^{-1}g \rightarrow g$ is a homomorphic mapping of the group H onto the factor-group G/Z. The kernel N of the mapping consists of the matrices of the form νE, with $\nu^n = 1$. Let the characteristic of the underlying field be p, $p > 0$, and suppose n is $p^k m$, $(m, p) = 1$. Then the relation $a^n = 1$ is equivalent to the relation $a^m = 1$, i.e., the orders of the elements of N divide m. We will show that the center of H is N.

Indeed, the central elements of an irreducible group H must be scalar matrices. Now a scalar matrix of H can come only from a scalar matrix of G. Hence, such a matrix must be of the form νE and be contained in N; i.e., N is the center of H.

Since H is solvable, the group H/N is also solvable and contains a non-trivial abelian normal divisor A/N. The group A is nilpotent since its commutator lies in the center of the subgroup N. Let A_m be the subgroup generated by the m-th powers of the elements of A. Since all elements of the commutator of A have order dividing m, it follows that A_m lies in the center Q of the group A. If Q is not contained in N, then Q has a central commutative normal divisor H. If we form the inverse image of Q in G, we obtain a non-central abelian normal divisor of G which contradicts the assumptions. Thus, $A_m \subset Q \subset N$. But the m-th power of an arbitrary element of A lies in the group A_m, a subgroup of N of order m. Therefore, the order of every element of A must be a divisor of the number m^2. According to the lemma, the group A must be finite, and consequently, its centralizer B must have finite index in H. The group A contains non-scalar matrices. All elements of B are permutable with these matrices; and hence, the group B is reducible. The inverse image of B in the group G is the required reducible subgroup of finite index.

It remains to note that the argument shows that the group mentioned in Theorem 1 has index not exceeding a certain bound depending only on the order of the matrices. In particular, we have the following result: the length of the chain of commutators of a solvable matrix group cannot surpass a certain bound depending only on the order of the matrices.

A group of triangular special matrices (diagonal elements 1) possesses the property that all characteristic numbers of all of its elements are 1. The converse is also true(Kuroš [4]): *if all characteristic numbers of every element of a matrix group over an algebraically closed field are* 1, *then the group is reducible to special triangular form.*

For the completeness, here is a brief proof of that remark. If the underlying field has characteristic $p > 0$, then if all characteristic numbers of the element a are equal to 1, it follows that the p^n power of a is the identity matrix. Thus, every finite set of elements of the given group generates a finite subgroup, the order of which is equal to p^k. By a known theorem, every such subgroup is reducible to special triangular form, and from this it follows immediately that the group can be transformed to special triangular form [6]. If the characteristic of the field is 0, let S be the free group with free generators a_g, where the index g runs through the given group G. Let us consider the representations $a_g \rightarrow g$ and $a_g \rightarrow e$ of the group S. The traces of the two representations coincide. Thus, it contains a corresponding equivalent irreducible term, so that the group G is reducible to special triangular form.

It is easy to see that if a group of matrices is reducible to special triangular form in the algebraic closure of the base field, then it is reducible in the base field since the requirement of algebraic closure is not an essential condition for the remark.

In the following, it is necessary to consider automorphisms of abelian groups of various types. Let M be an abelian group of finite rank without elements of finite order written additively. Let the elements $m_1, \cdots, m_s$ be a maximal system of linearly independent elements in M. Then every element m of M has a multiple tm (not 0) and with the property that the relation

$$tm = t_1 m_1 + \ldots + t_s m_s \tag{1}$$

holds with suitable t_i. We write the relation (1) in the equivalent form

$$m = \frac{t_1}{t} m_1 + \ldots + \frac{t_s}{t} m_s.$$

Thus, every element of M can be written as a linear combination of the m_i with rational coefficients, and this representation is unique. Thus, the group M is included in the group M^* of linear forms in the m_i, and possesses, thus, the property that some multiple of an arbitraty element of M^* belongs to M. M^* has no element of finite order, and the equation $tx = m$ is always solvable in M^* when t is an arbitrary positive integer. Let σ be an automorphism of the group M. Then we can write $m_i^\sigma = a_{i1} m_1 + \cdots + a_{is} m_s$, where a_{ij} are certain rational numbers. The correspondence $\sigma \longrightarrow (a_{ij})$ allots to an automorphism of the group M a non-singular matrix with rational elements. On the other hand, not every matrix gives an automorphism of the group M in general.

Lemma 2. *Let G be the group of automorphisms of an abelian torsion-free group M of finite rank. If the matrices of G are simultaneously reducible in the field of rational numbers to triangular form, then the group M has a chain of subgroups $M \supset M_1 \cdots \supset M_{s-1} \supset 0$ invariant relative to G, and with factor groups of rank one and without elements of finite order.*

Let M be written additively and form the group M^* associated with M and defined above. M^* can be considered as a linear space over the field of rational numbers. By hypothesis there is a chain of invariant subspaces $M^* \supset M_1^* \supset \cdots \supset M_{s-1}^* \supset 0$ of dimensions $s, s-1, \cdots, 1, 0$ in M^*. The set of all the elements of M which have multiples in M_i^* is a subgroup M_i of the group M, invariant with respect to the automorphisms of G, and the chain M_i satisfies the conditions of the theorem.

Now let a given abelian group M be decomposed into the direct sum $M = M_1 \dotplus \ldots \dotplus M_s$, where each sum and M_i is a group of type p^∞, p being a fixed prime number. Denote by $a_j^{(i)}$ $(j = 1, 2, \cdots)$ a system of generators of M_i satisfying the relations $pa_{j+1}^{(i)} = a_j^{(i)}$. Let σ be an endomorphism of M, and suppose

$$a_j^{(i)\sigma} = \alpha_{j1}^{(i)} a_j^{(1)} + \alpha_{j2}^{(i)} a_j^{(2)} + \ldots + \alpha_{js}^{(i)} a_j^{(s)},$$

where $a_{jk}^{(i)}$ are integers mod p^j and connected by the relation

$$pa_{jk}^{(i)} \equiv a_{j-1,k}^{(i)} (\operatorname{mod} p^{j-1}).$$

Thus the sequence

$$\alpha_k^{(i)} = \{\alpha_{1k}^{(i)}, \ \alpha_{2k}^{(i)}, \ldots, \ \alpha_{nk}^{(i)}, \ldots\}$$

may be considered as an integral p-adic number, and the endomorphism σ gives rise to the matrix $(a_k^{(i)})$ of degree s with integral p-adic elements $a_k^{(i)}$. Moreover, every such matrix corresponds to some endomorphism of the group M. The product of two endomorphisms corresponds to the product of the corresponding matrices. Moreover, for an endomorphism to be an automorphism, it is necessary and sufficient that its matrix have an integral inverse, that is, that its determinant be not divisible by p.

Lemma 3. *If matrices of some group of automorphisms G of the direct sum M of groups of type p^∞ are simultaneously reducible in the field of p-adic numbers to triangular form, then M contains subgroups $\overline{M}_i$ of type p^∞, where M is the direct sum $\overline{M}_1 \dotplus \cdots \dotplus \overline{M}_s$, and the subgroups $\overline{M}_k + \overline{M}_{k+1} + \cdots + \overline{M}_s$ are invariant under the automorphisms of G.*

The matrices of the automorphisms of G have integral p-adic elements and determinant not divisible by p. Suppose that all the matrices of G can be simultaneously transformed into triangular form, where the transforming matrix has integral p-adic elements, with determinant possibly divisible by p. It is necessary to find a matrix having the same property, but with determinant not divisible by p. Let E^* be the auxiliary space with respect to the field of p-adic numbers with basis $e_1, e_2, \cdots, e_s$, and let E be the set of vectors in E^* having integral coordinates with respect to this basis. The matrices in G transform E into itself. It is required to find in E a new basis, with respect to which the matrices of G have triangular form. By hypothesis, there is in E^* a vector

$$u = \alpha_1 e_1 + \ldots + \alpha_s e_s,$$

satisfying the relation

$$u^\sigma = \Lambda(\sigma) u \quad (\sigma \in G).$$

Let us seek a t such that the coordinates of tu are integers and such that at least one of its coordinates, say the first, is not divisible by p. Then $tu, e_2, \cdots, e_s$ are a new basis for E. The subspace $\{\lambda u\}$ is invariant under G, and we can consider the quotient space $G/\{\lambda u\}$. By repeating this process, we obtain a basis for E in which the matrices of the automorphisms of G are in triangular form. The change from the first to the second basis involves a transformation, the matrix T of which has integral p-adic elements and determinant not divisible by p. Thus, T is the matrix of some automorphism τ of the group M. We have supposed that the group M is a direct sum ΣM_i of subgroups of type p^∞. But since the matrices of G have triangular form, it follows that the space $M = \Sigma M_i^{\tau}$ and the subspaces

$$M_k^{\tau} + M_{k+1}^{\tau} + \ldots + M_s^{\tau}$$

are invariant under G.

Theorem 2. *Every solvable group of automorphisms of an abelian group M with a finite number of generators has a finite chain of normal subgroups with cyclic factors.*

Proof. Let us denote the periodic subgroup of the group M by P. Every automorphism σ of M induces an automorphism $\bar{\sigma}$ in the group M/P. The correspondence $\sigma \rightarrow \bar{\sigma}$ is a homomorphism with finite kernel. Thus, it is enough to prove the theorem in the case that M has no elements of finite order. Then M is the direct product of a finite number of infinite cyclic groups, and the group of automorphisms of M is isomorphic to the group of unimodular matrices.

Now let G be a solvable group of matrices with integral elements, determinant ± 1, of degree s. By Theorem 1, G has a subgroup H of finite index, the matrices of which can be transformed into triangular form in a suitable extension K of the field of rational numbers. By the proof of Theorem 1, the field K can be taken to have finite degree. The elements of the matrix X which transforms G into triangular form can be taken to contain integral elements of the field K. Then if we multiply the elements of all matrices of the system $X^{-1}HX$ by the same integer, for example, by the determinant of the matrix X, we make them integers; that is, the elements of the matrix $X^{-1}HX$ have a common denominator.

The diagonal elements of the matrix $X^{-1}AX$, for A in H, are the same as the characteristic numbers of A. Since the determinant of the matrix A is equal to ± 1, its characteristic numbers are units in the field K. If we put down for every element A of H the corresponding diagonal numbers of the matrix $X^{-1}AX$, we obtain a representation of the group H, the kernel H_0 of which consists of all matrices of the system $X^{-1}HX$ with diagonal elements equal to 1. By a theorem of Dirichlet, the group of units of the field K has a finite number of generators. Thus, the group D which consists of all the diagonal matrices, the diagonal elements of which are units from K, also has a finite number of generators. The factor group H/H_0 is isomorphic to some subgroup of the group D, and therefore, has itself a finite number of generators. Now let us consider H_0. This group consists of triangular matrices with 1's on the diagonal; moreover, the elements of all its matrices have a common denominator. Let H_{0i} be the set of all matrices of the group H_0 in which the first i lines parallel to the main diagonal consist of zeros. The chain $H_0 \supset H_{01} \supset \cdots \supset H_{0,\, s-1} = 1$ is a normal series of the group H_0. The factor-group $H_{0i}/H_{0,\, i+1}$ is isomorphic to a certain subgroup of the direct sum of additive groups of the field K, which has denominator det X.. Since the additive group of the field K consisting of fractions with given denominator has a finite number of generators, the factor group $H_{0i}/H_{0,\, i+1}$ must have a finite number of generators.

§2. Solvable A_i-groups

To study the properties of solvable groups, we impose various natural properties on the factors of a resolvent chain of normal subgroups beyond the commutative character of the factor groups. Among the classes of commutative groups, the simplest are the following: A_1) commutative groups such that the factor group with respect to the periodic part has finite rank; A_2) groups of type A_1, the periodic part of which can be decomposed into the direct product of cyclic and locally cyclic groups; A_3) groups of type A_1, the periodic part of which can be decomposed into the direct product of a finite number of cyclic and *primary* locally cyclic subgroups (of type p^∞); A_4) groups of type A_1, the periodic part of which is finite; A_5) abelian groups with a finite number of generators. Correspondingly, a group will be called a *group of type* A_i if it has a finite chain of normal subgroups, every factor of which is an abelian group of type $A_i (i = 1, 2, 3, 4, 5)$.*

Making use of the fact that a subgroup of an abelian A_i-group is an A_i-group, $(i = 1, \cdots, 5)$, we can arrive at the conclusion that a *subgroup of a solvable A_i-group is a solvable A_i-group.*

Similarly, for $i = 1, 2, 5$, a factor group of an abelian A_i-group is itself an A_i-group. Thus, *a factor group of a solvable A_i-group is again a solvable A_i-group* $(i = 1, 2, 5)$. For $i = 3, 4$ in general it can only be asserted that the factor group of a solvable A_i-group with respect to a *periodic* normal divisor is a solvable A_i-group.

If we call the *reduced rank* of an abelian group the rank of the factor group with respect to the subgroup of all periodic elements, it is easy to show by the use of Schreier's theorem that *the sum of the reduced ranks of the factors of a solvable normal chain of an A_1-group does not depend on the choice of the chain.*

The product of periodic normal divisors of an arbitrary group is again a periodic normal divisor. Thus, every group has a unique maximal periodic normal divisor.

Theorem 3. *The factor group of a solvable A_1-group with respect to its maximal periodic normal divisor is a solvable A_4-group, having a finite chain of characteristic subgroups with abelian factors of type A_4.*

Proof. A solvable group is characterized by the property that in the series of successive commutators, the identity appears after a finite number of steps. We prove Theorem 3 by induction on the number of the term which is equal to the identity. If the first commutator is 1, the group is commutative, and the theorem is obvious. Let us suppose that the theorem is proved in all cases when the identity is the n-th commutator, and consider a certain A_1-group G for which the identity is the $(n + 1)$-th commutator $G^{(n+1)}$. Let P be the maximal periodic normal divisor of G, and let $H = G/P$, so that H is a solvable A_1-group with no periodic divisor except 1, and with $H^{(n+1)} = 1$. $H^{(n)}$ is a characteristic commutative subgroup of the group H. Since the maximal periodic normal divisor of H is 1, $H^{(n)}$ contains no elements of finite order, and thus has an abelian normal torsion-free divisor of finite rank. Let $Q/H^{(n)}$ be a maximal periodic normal divisor of the factor-group $H/H^{(n)}$. Since the n-th commutator of this factor group is 1, then it is itself of type A_1, and by induction hypothesis the group

$$H/H^{(n)}/Q/H^{(n)} \cong H/Q$$

has a chain of characteristic subgroups with abelian factors of type A_4.

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* The properties of A_5-groups were considered by Hirsch in [3], where such groups were called *S*-groups.

The subgroup $H^{(n)}$ is a normal divisor in Q. The inner automorphisms of Q induce in $H^{(n)}$ automorphisms which, with respect to a definite basis for $H^{(n)}$(see §1), can be represented by matrices over the field of rational numbers. The corresponding representation D of the group Q has for a kernel the centralizer of the subgroup $H^{(n)}$ in the group Q. Thus, D is isomorphic to the factor group Q/Z, which may thus be considered as a group of matrices over the field of rational numbers. Since the group $Q/H^{(n)}$ is periodic and $H^{(n)}$ is contained in Z, then Q/Z is also a periodic group. But every periodic group of matrices over the field of rational numbers is finite. Thus, the group Q/Z is finite.

Now consider the group Z. The subgroup $H^{(n)}$ is central in the group $H^{(n)}Z^{(n-1)}$ with factor group $H^{(n)}Z^{(n-1)}/H^{(n)}$ which is commutative. Thus, the group $H^{(n)}Z^{(n-1)}$ is nilpotent. This group contains no periodic elements except 1, since the periodic elements of a nilpotent group form a characteristic subgroup, which would have to be a periodic normal divisor of the same group H which by hypothesis has no periodic normal divisor except 1. Since the group

$$H^{(n)}Z^{(n-1)}/H^{(n)}$$

is periodic, the group $H^{(n)}Z^{(n-1)}$ is the *complement* of the group $H^{(n)}$ in the sense of [8]. But the complement of an abelian group is abelian, and thus the group $H^{(n)}Z^{(n-1)}$ is commutative. Since $Z^{(n-1)}$ is commutative, it follows that $Z^{(n)}$ is 1. Thus, the group Z satisfied the induction hypothesis and has a finite chain of characteristic subgroups with commutative factors of type A_4.

Now, the group H has characteristic subgroups $H \supset Q \supset Z \supset 1$ such that the factor-groups H/Q, Q/Z, Z have a finite chain of characteristic subgroups with abelian factors of type A_4. It follows necessarily that this must be true of the group H.

By definition, a solvable A_i-group is a group which contains a normal chain of subgroups with abelian factor groups of type $A_i (i = 1, 2, 3, 4, 5)$. From Theorem 3, it can be shown that in that definition the normal chain can be taken to be a chain of characteristic subgroups. We have the corollary

Corollary. Every solvable A_i-group $(i = 1, 2, 3, 4, 5)$ has a finite chain of characteristic subgroups with abelian factors of type A_i.

Since a factor group of a solvable A_i-group $(i = 1, 2, 5)$ is itself an A_i-group, then for those values of i the chain of successive commutators is a chain with the required properties.

Let G be a solvable group of type A_4. Let P be the maximal periodic normal divisor of the group G. The group P is finite. By Theorem 3, the factor group G/P has a finite chain of characteristic subgroups, the factors of which are commutative and have type A_4. The inverse images in G of the chain of successive commutators, together with the group P itself form the required chain of characteristic subgroups of the group G.

Last, let G be a solvable group of type A_3, and let P be its maximal periodic normal divisor. Since P is a periodic solvable A_3-group, and since a factor group of a periodic solvable A_3-group is itself an A_3-group, then the sequence of successive commutators of P is a chain of characteristic subgroups with abelian factors of type A_3. If we take the chain of inverse images of the subgroups of a characteristic chain of the factor group G/P, we obtain again a chain of characteristic subgroups of the group G.

Theorem 4. *Every solvable group G of type A_3 has a subgroup of finite index with nilpotent commutator.*

Proof. By Theorem 3 and its corollary, the group G has a chain of normal divisors $G \supset G_1 \supset \cdots \supset G_n \supset 1$, the factor groups of which are either finite, or torsion-free abelian groups of finite rank, or decomposable into the direct product of a finite number of abelian groups of type p^∞(for one and the same p). In the group G we construct a descending chain of subgroups H_i with the following properties: take $H_0 = G$ and proceed by induction. Suppose the subgroup H_i already constructed. The inner automorphisms of the group G induced by the elements of H_i induce certain automorphisms of the factor group G_{i-1}/G_i. If this factor group is finite, then H_{i+1} denotes the set of elements of H_i which induce the identity automorphism in G_{i-1}/G_i. In this case, H_{i+1} must have finite index in H_i since G_{i-1}/G_i is finite. Next, let G_{i-1}/G_i be a torsion-free group or the direct product of a finite number of groups of type p^∞. Then the automorphisms induced by the elements of H_i can be (§1) represented by matrices over the field of rational or p-adic numbers. Denote by D_i the set of matrices into which the elements of H_i are mapped. D_i is a solvable group of matrices, and so by Theorem 1, D_i contains a subgroup F_i of finite index, the matrices of which can be transformed into triangular form in the algebraic closure of the base field. The set of all elements of H_i which are mapped into F_i is a subgroup of finite index in H_i. This subgroup is chosen for H_{i+1}. Let the last subgroup H_{n+1} be denoted by H. Since all the indices $H_{i-1}:H_i$ $(i = 1, 2, \cdots, n + 1)$ are finite, the group H must be a subgroup of finite index in G. It remains to show that H' is nilpotent.

The automorphisms induced by the elements of H fix the elements of finite order in G_{i-1}/G_i and can be represented as triangular matrices with infinite factor group. But if these elements of H are represented as triangular matrices, the elements of the commutator H' can be represented as triangular matrices with 1's on the diagonal since by Theorem 1, the representation of H' can be reduced to triangular form in the base field. In view of Lemmas 2 and 3, it is clear that every infinite factor group G_{i-1}/G_i has a chain of subgroups invariant relative to the elements of H'

$$G_{i-1} / G_i \supset B_{i1} / G_i \supset \cdots \supset B_{ik_i} / G_i \supset G_i / G_i,$$

such that the factor groups of two successive terms have rank 1, and the elements of H' induce only the identical automorphism in them. In the chain $G \supset G_1 \supset \cdots \supset G_n \supset 1$, if whenever neighboring subgroups G_{i-1}, G_i have infinite factor group G_{i-1}/G_i, we insert the intermediate groups $B_{i1}, \cdots, B_{ik_i}$, we obtain a new chain $G \supset K_1 \supset K_2 \supset \cdots \supset K_s \supset 1$ of subgroups of G. The subgroups K_i are invariant with respect to the elements of H', and the latter induces in the factor groups K_{i-1}/K_i only the identical automorphism. Form the groups

$$N_i = H' \cap K_i \; (i = 1, 2, \cdots, s).$$

The chain $H' \supset N_1 \supset \cdots \supset N_s \supset 1$ consists of normal divisors of the group H'.

The elements of H' induce the identical automorphism on the factor groups N_i/N_{i+1}. Thus, the last chain is a descending central series for the group H', and H' is nilpotent.

§3. Locally nilpotent groups

We call a group G *locally nilpotent* if every finite set of elements lies in some nilpotent subgroup of the group G. Similarly, a group is called *locally solvable* if every finite set of elements of G lies in some solvable subgroup. It is of interest to see what conditions can be imposed on the subgroups of the group G so that solvability or nilpotency follows from local solvability or local nilpotency.

Theorem 5. *If every abelian subgroup of a locally nilpotent group G is of type A_1, and if P is the maximal periodic normal divisor of the group G, then the factor group G/P is a nilpotent group of type A_4.*

Proof. The collection of all elements of finite order of a locally nilpotent group is a normal divisor of the group. Thus, the factor group G/P has no element of finite order except 1. Let us show that all abelian subgroups of the group G/P have type A_1, i.e., are torsion-free abelian groups with finite rank.

Suppose A/P to be an abelian subgroup of G/P and of infinite rank. Let us choose an infinite set of elements $a_1, a_2, \cdots$ in A which have linearly independent images in A/P. The subgroup H_i generated by the elements a_1, a_i in A is a nilpotent group with two generators. Thus, the periodic part P_i of this group is finite. Moreover, the automorphism induced in P_i by the element a_i has finite order n_i. Since the factor group H_i/P_i is finite, the relations

$$a_i^{-n_i} a_1 a_i^{n_i} = a_1 c_i \ \ (c_i \in P_i); \qquad a_i^{-kn_i} a_1 a_i^{kn_i} = a_1 c_i^k$$

hold ($k = 1, 2, \cdots$). Since c_i has finite order m_i, this leads to the relation

$$a_i^{n_i m_i} a_1 = a_1 a_i^{n_i m_i}.$$

Thus, the element a_1 is commutative with the subgroup generated by the $(n_i m_i)$-th power of the element a_i. By the same kind of argument, it can be seen that the element $a_2^{n_2 m_2}$ is commutative with a certain finite power $a_i^{n_i m_i k_i}$ of the element $a_i^{n_i m_i}$ ($i = 3, 4, \cdots$), etc. The system $a_1, a_2^{n_2 m_2}, a_3^{n_3 m_3 k_3}, \cdots$ generates an abelian subgroup of infinite rank in G, which contradicts the hypothesis of the theorem. Thus, the group G/P is locally nilpotent, contains no periodic elements, and all its abelian subgroups have finite rank. By [8], it follows that the group G/P can be imbedded in a complete locally nilpotent group G^* which has no periodic elements and such that a suitable finite power of every element of G/P is contained in G^*. Every complete abelian normal subgroup of G^* has a complement corresponding to an abelian subgroup of G/P. Since the rank of an abelian group is the same as that of its complement, the group G^* is a locally nilpotent complete group without elements of finite order, and every abelian subgroup has finite rank. Let $\mathfrak{G}$ be the Lie algebra of the group G^*. By [8], the algebra $\mathfrak{G}$ is locally nilpotent, and every abelian subalgebra has finite rank. It remains to show that $\mathfrak{G}$ is nilpotent.

The following reasoning is based on analogous lemmas of Hall and originally on parallel reasoning of H. H. Muhammedžan [10] and S. N. Černikov [13].

Lemma 4. *If the Lie algebra $\mathfrak{G}$ has an ascending central series*

$$\mathfrak{Z}_1 \subset \mathfrak{Z}_2 \subset \cdots \subset \mathfrak{Z}_n \subset \cdots, \quad \bigcup \mathfrak{Z}_n = \mathfrak{G},$$

with factors of finite rank, then the series breaks off in a finite number of steps.

By definition, an ascending central series is such that $\mathfrak{Z}_0$ is 0, and $\mathfrak{Z}_n$ is the set of elements a of the algebra $\mathfrak{G}$ for which $[a, \mathfrak{G}]$ lies in $\mathfrak{Z}_{n-1}$. Let $\mathfrak{A}$ be an ideal of the algebra $\mathfrak{G}$. Then if for some j, the relation $\mathfrak{A} \cap (\mathfrak{Z}_{j+1} \setminus \mathfrak{Z}_j) \neq \Lambda$ holds, where Λ is the empty set, then for arbitrary i less than j, also the relation $\mathfrak{A} \cap (\mathfrak{Z}_{i+1} \setminus \mathfrak{Z}_i) \neq \Lambda$ holds. Suppose that a is an element in both $\mathfrak{A}$ and $\mathfrak{Z}_{j+1} \setminus \mathfrak{Z}_j$. Suppose it were true that for every g in $\mathfrak{G}$, the commutator $[a, g]$ were not in $\mathfrak{Z}_j \setminus \mathfrak{Z}_i$; then, it would follow from the relations $[a, g] \in \mathfrak{A}$, $[a, g] \in \mathfrak{Z}_j$ that $[a, g]$ would have to lie in $\mathfrak{Z}_i$; that is, a would lie in $\mathfrak{Z}_{i+1}$, which contradicts the supposition. Thus, it is true for every i less than j that there is an element g for which $[a, g]$ lies in $\mathfrak{Z}_j \setminus \mathfrak{Z}_i$; and thus, in particular, the relation

$$\mathfrak{A} \cap (\mathfrak{Z}_j \setminus \mathfrak{Z}_{j-1}) \neq \Lambda$$

holds.

From the above it follows that every ideal $\mathfrak{A}$ of the algebra $\mathfrak{G}$ is either contained in some $\mathfrak{Z}_j$ or else has non-empty intersection with each difference $\mathfrak{Z}_{j+1} \setminus \mathfrak{Z}_j$.

Let us prove the lemma. Let $\mathfrak{N}_i$ be the centralizer of $\mathfrak{Z}_i$ in $\mathfrak{G}$; that is, the set of elements g in $\mathfrak{G}$ for which $[g, \mathfrak{Z}_i]$ is 0. The $\mathfrak{N}_i$ form a descending chain of ideals in $\mathfrak{G}$; $\mathfrak{G}$ is $\mathfrak{N}_1$; the intersection of the $\mathfrak{N}_i$ is $\mathfrak{Z}_1$. By hypothesis, the rank of the algebra $\mathfrak{Z}_i$ is finite. Thus, the rank of the quotient algebra $\mathfrak{G}/\mathfrak{N}_i$ is also finite. Now let us show that the algebra $\mathfrak{G}$ has finite rank. If for some natural numbers p, q, the relation $\mathfrak{N}_p \subset \mathfrak{Z}_q$ holds, then the rank of $\mathfrak{G}/\mathfrak{Z}_q$ cannot be more than the rank of $\mathfrak{G}/\mathfrak{N}_p$ and, thus, is finite. If $\mathfrak{N}_p$ is contained in no $\mathfrak{Z}_q$, whatever p, q, then by the above, the intersection $(\mathfrak{Z}_2 \setminus \mathfrak{Z}_1) \cap \mathfrak{N}_p$ is not empty for any p. The subalgebra $\mathfrak{N}_p \cap \mathfrak{Z}_2$ is not $\mathfrak{Z}_1$, but contains $\mathfrak{Z}_2$. The rank of $\mathfrak{N}_p \cap \mathfrak{Z}_2$ cannot be unbounded. Hence, for some p, the relations $\mathfrak{N}_p \cap \mathfrak{Z}_2 = \mathfrak{N}_{p+1} \cap \mathfrak{Z}_2 = \cdots$ must hold. But the intersection of these last sets $\mathfrak{N}_{p+i} \cap \mathfrak{Z}_2$ is contained in $\mathfrak{Z}_1$; that is, $\mathfrak{N}_p \cap \mathfrak{Z}_2$ is contained in $\mathfrak{Z}_1$; and this contradicts the supposition that the intersection $(\mathfrak{Z}_2 \setminus \mathfrak{Z}_1) \cap \mathfrak{N}_p$ is not empty.

Lemma 5. *If the Lie algebra $\mathfrak{G}$ is locally nilpotent, and if all its commutative subalgebras have finite rank, then the algebra $\mathfrak{G}$ is nilpotent and has finite rank.*

Suppose $\mathfrak{G}$ is not nilpotent. Then $\mathfrak{G}$ has an ascending chain of nilpotent subalgebras $\mathfrak{H}_1 \subset \mathfrak{H}_2 \subset \cdots$, the length of the central series of which tends to infinity. Let $\mathfrak{A}_i$ be the center of the algebra $\mathfrak{H}_i$. The sum $\mathfrak{A}_1 + \mathfrak{A}_2 + \cdots$ is an abelian subalgebra of $\mathfrak{G}$, and therefore has finite rank. It follows that for some i, the relation $\mathfrak{A}_i = \mathfrak{A}_{i+1} = \cdots$ must hold; and therefore, $\mathfrak{A}_i$ is the center of the algebra $\mathfrak{H}$, defined by $\mathfrak{H} = \bigcup \mathfrak{H}_i$. Indeed, the algebra $\mathfrak{H}$ has a non-trivial center, which we denote by $\mathfrak{Z}_1$. Let us show that every abelian subalgebra of $\mathfrak{H}/\mathfrak{Z}_1$ has finite rank. Indeed, if $\mathfrak{H}/\mathfrak{Z}_1$ contains an abelian subalgebra $\mathfrak{F}/\mathfrak{Z}_1$ of infinite rank, let $a_1, a_2, \cdots$ be elements of the algebra $\mathfrak{F}$ which have linearly independent images in $\mathfrak{F}/\mathfrak{Z}_1$. If $[a_1, a_i]$ be denoted by z_i, then z_i is in $\mathfrak{Z}_1$. Since the algebra $\mathfrak{Z}_1$ has finite rank, then we can find an infinite set of linearly independent elements $a_2', a_3', \cdots$ which are linear combinations of $a_2, a_3, \cdots$ and such that the commutators $[a_1, a_i']$ are all 0. If we proceed similiarly with $a_2', a_3', \cdots$, we obtain an infinite sequence of elements a_1, a_2', a_3'', which are commutative and linearly independent in $\mathfrak{H}$, which contradicts the hypothesis. Thus, $\mathfrak{H}/\mathfrak{Z}_1$ is countable, locally nilpotent, and all its abelian subgroups have finite rank. By what has already been proved, the quotient $\mathfrak{H}/\mathfrak{Z}_1$ has a non-trivial center $\mathfrak{Z}_2/\mathfrak{Z}_1$. By the same argument, the quotient $\mathfrak{H}/\mathfrak{Z}_2$ has a non-trivial center $\mathfrak{Z}_3/\mathfrak{Z}_2$, etc. Let $\mathfrak{Z}$ be the union $\bigcup \mathfrak{Z}_i$ of all the $\mathfrak{Z}_i$. The algebra $\mathfrak{Z}$ has an ascending central series $\mathfrak{Z}_1, \mathfrak{Z}_2, \cdots$. It follows that $\mathfrak{Z}$ has an upper central series. Since the rank of every abelian subalgebra of $\mathfrak{Z}$ is finite, every factor of its ascending central series also has finite rank. By Lemma 4, it follows that $\mathfrak{Z}$ has finite rank. Thus, the sequence $\mathfrak{Z}_1, \mathfrak{Z}_2, \cdots$ is finite, and $\mathfrak{H}$ is nilpotent. But that contradicts the hypothesis that $\mathfrak{H}$ contains subalgebras $\mathfrak{H}_i$, the length of the upper central series of which is unbounded. Lemma 5, and thus, Theorem 5, are proved.

Let G be a locally nilpotent group such that every abelian subgroup is of type A_3. Let P be the periodic part of the group G. By Theorem 5, the factor group G/P is nilpotent of type A_3. On the other hand, P is a locally nilpotent group, every abelian subgroup of which satisfies the descending chain condition for subgroups. By a result of S. N. Černikov [13], P is solvable and of type A_3. Since the quotient group G/P and P itself are solvable A_3-groups, the group G must also be a solvable A_3-group. Thus, *if every abelian subgroup of a locally nilpotent group* G *is of type* A_3, *then* G *is a solvable* A_3*-group.*

Theorem 5 has the following corollary.

Corollary. If every abelian subgroup of a locally nilpotent group G is of type A_4, then G is a nilpotent group.

Indeed, by hypothesis, every abelian subgroup of the periodic part P of the group G is finite. Since P is nilpotent, P is finite. By obvious considerations, it is easy to see that every finite normal divisor $(\neq 1)$ of a locally nilpotent group contains at least one non-trivial central element of the group. Thus, if $1 \subset G_1 \subset G_2 \subset \cdots$ is the upper central series of the group G, then for some finite number m, the group P is contained in G_m; and since by Theorem 5 the factor-group G/P is nilpotent, there must be some finite n for which G_n is G, and the assertion is proved.

Theorem 6. *If every finite set of elements of the group* G *lies in some nilpotent normal divisor of* G, *and if all abelian subgroups of the group* G *are of type* A_3, *then* G *is a nilpotent* A_3*-group.*

Proof. Let Q be the maximal periodic divisor of the group G. Let Q^∞ be the set of elements of Q which are infinitely divisible. Q^∞ is an abelian normal divisor of the group G, and the index of Q^∞ in Q is finite. Moreover, every abelian subgroup of the factor group G/Q^∞ is of type A_4 and so by the corollary to Theorem 5, the factor group must be nilpotent. The group Q^∞ is a direct product of a finite number of its primary subgroups $Q_1, Q_2, \cdots, Q_s$, each of which is in turn the direct product of a finite number of groups of type p^∞.

Thus, to prove the theorem, it will be enough to prove the following assertion. If every finite set of elements of the group G is contained in some nilpotent normal divisor of the group G, and if G has a normal divisor H which can be decomposed into the direct product of a finite number of subgroups of type p^∞, and if the factor group G/H is nilpotent, then G is nilpotent.

First, let us consider the case when H is decomposed into the direct product of a finite number of factors of type p^∞ for one and the same prime number p. Let a be an arbitrary element of the group G; let A be the automorphism worked on H by transforming the elements of H by a. Write $a^{-1}ha = hA$, $h \in H$; also $(h, a) = h^{-1}a^{-1}ha = h(A - E)$, where E is the identical automorphism. By hypothesis, the element a lies in some nilpotent normal divisor N of the group G. Therefore, (h, a) lies in H and $h(A - E)$ lies in N. But since N is nilpotent, it follows that for suitable positive n the relation $((\cdots (f, a)a) \cdots)a) = 1$ holds for arbitrary f in N, and thus, when f is (h, a), the relation $h(A - E)^{n+1} = 0$ holds, so that $(A - E)^{n+1}$ is 0. This means that every characteristic value of the matrix corresponding to the transformation A is 1. If we make correspond to every element a of G the transformation A which it induces in H, we obtain a representation of the group G.

When H can be decomposed into the direct sum of groups of type p^∞, the representation A can be taken as a matrix over the field of p-adic numbers. The set of matrices which represent the elements of the group G also form a group, and all characteristic numbers of these matrices are equal to 1. But by §1, we can conclude that all matrices can be simultaneously transformed into triangular form; and by Lemma 4, there must be a system H_i of subgroups in H which are invariant under transformation by A, which form an ascending set from 1 to H, and such that the quotient H_{i+1}/H_i is a group of type p^∞. Therefore, H_i is a normal divisor of the group G. Since the characteristic roots of the transformation A are equal to 1, the automorphism induced on the factor group H_{i+1}/H_i by an inner automorphism of G is the identical automorphism. Thus, the system H_i of subgroups is an ascending central series of G. If we add the subgroups which compose an upper central series for G/H, we obtain the required central series of the group G, ending with G.

The proof of the theorem in the general case is based on induction on the number of primary components into which the group H can be decomposed. Let the given group G have a normal divisor H which can be decomposed into s primary components, and let the theorem be proved for normal divisors with a smaller number of primary components. Let p be a prime number which corresponds to a non-trivial primary component Q_p. Let Q be a subgroup of the group G corresponding to the remaining prime numbers. The factor group G/Q has a primary normal divisor H/Q, and satisfies all hypotheses for the case for which the theorem is already proved. In view of the induction hypothesis, we can assume that the group G/Q is nilpotent. Thus, the group G has a normal divisor Q with nilpotent factor group G/Q and satisfying the other requirements of the assertion. Since Q has at most $s - 1$ primary components, then the group G must be nilpotent by induction.

Theorem 6 can be applied to the question of the existence of a maximal nilpotent normal divisor. Kuroš showed that the product of any two nilpotent normal divisors of an arbitrary group is a nilpotent normal divisor of that group. Thus, the product of all the nilpotent normal divisors of an arbitrary group has also the property that an arbitrary finite set of elements in this product is contained in some nilpotent normal divisor. If we suppose that all abelian subgroups of a given group G are of type A_3, then the product of its normal divisors possesses the same property. In view of Theorem 6, this product must be nilpotent and is, moreover, the maximal nilpotent normal divisor of the group G. Thus, *if every abelian subgroup of a group G is of type A_3, then G has a maximal nilpotent normal divisor.*

In particular, if G is a solvable A_3-group, then G contains a maximal nilpotent normal divisor N. Theorem 4 shows that there is a normal divisor H of finite index in G, the commutator H' of which is nilpotent. Since H' is a nilpotent normal divisor of the group G, H' must be contained in N, and we see that the factor group G/N is a finite extension of an abelian subgroup. In a similar manner, we can consider the question of the existence of a maximal solvable normal divisor of an arbitrary group.

Theorem 7. *Suppose that every finite set of elements of the group G is contained in some solvable normal divisor and suppose that every abelian subgroup of the group G is of type A_3. Then the group G is solvable.*

Proof. If the lengths of the series of commutators of the solvable normal divisors of G are bounded, then because of the local hypothesis, G is solvable. Thus, we can assume that G has a set of ascending solvable normal divisors H_i, with increasing lengths for the chain of commutators. Let us consider the group $\bigcup H_i$. Its abelian subgroups are of type A_3. Thus, H contains a maximal nilpotent normal divisor N, which by Theorem 6 is of type A_3. Let $1 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_s = N$ be an ascending central series of the group N; this series consists of characteristic subgroups, and suppose that every factor group N_{i+1}/N_i is an abelian torsion-free group of finite rank, or a direct product of a finite number of groups of type p^∞, or a finite abelian group. The inner automorphisms H generated by an arbitrary element a induce some automorphism θ_i on each factor group N_{i+1}/N_i. By taking a given basis for N_{i+1}/N_i, the automorphism θ_i can be represented as a matrix over a suitable field. If the characteristics of the fields are all 0, then they can be taken to be subfields of one and the same field. If we put the element a into correspondence with the direct sum of the matrices θ_i, we obtain a homomorphic mapping of the group H into a group of matrices. Let us denote the kernel of that mapping by R. The factor group H/R is locally solvable and is isomorphically represented by the matrices. By Theorem 1, it follows that every locally solvable group of matrices is solvable. Thus, H/R is solvable. If we can show that R is contained in N, then all is proven since H is solvable as soon as N is nilpotent and H/N is solvable.

Suppose that NR is not equal to N; then for some m, the relation

$$NR \cap NH_m \neq N$$

must hold. Call the intersection of NR with NH_m, D. D is a solvable normal divisor of H, and for some member $D^{(i)}$ of its commutator series, the relations

$$D^{(i)} \subset N, \ D^{(i-1)}N \neq N$$

must hold. $D^{(i-1)}N$ is a solvable normal divisor of H, greater than N. The chain

$$D^{(i-1)}N \supset N \supset N_{s-1} \supset \cdots \supset N_1 \supset 1$$

is a descending central series for $D^{(i-1)}N$, and we have the conclusion that N is not a maximal nilpotent normal divisor of H. This proves Theorem 7.

The following corollary can be obtained:

Corollary. If every abelian subgroup of the group G is of type A_3, then G has a maximal solvable normal divisor.

Indeed, let R be the product of all solvable normal divisors of G. Since the product of a finite number of solvable normal divisors is again a solvable normal divisor, it follows that any finite number of elements of R is contained in some solvable normal divisor of R. It is also given that all abelian subgroups of R are of type A_3. By Theorem 7, the subgroup R is solvable; and thus, it is the maximal solvable normal divisor of G.

In conclusion, we prove the following assertion:

Theorem 8. *If every abelian subgroup of the solvable group G is of type A_5, i.e., has a finite number of generators, then the group G is of type A_5.*

Proof. We use induction on the length of the chain of commutators. Let the $(n+1)$-th commutator $G^{(n+1)}$ of the given group be 1, and let the theorem be assumed true for any group which has n-th commutator equal to 1. By hypothesis of the theorem, $G^{(n)}$ is an abelian group with a finite number of generators. Let H be the centralizer of $G^{(n)}$ in G. H is isomorphic to some subgroup of the group of automorphisms of $G^{(n)}$ and, in view of Theorem 2, H is an A_5-group. As just pointed out, the normal divisor $G^{(n)}$ is also an A_5-group. Thus, if we can show that the factor group $H/G^{(n)}$ is an A_5-group, the theorem will be proved. Let $A/G^{(n)}$ be an abelian subgroup of the group $H/G^{(n)}$. Let us consider an arbitrary maximal abelian subgroup B of the group A. Since $G^{(n)}$ lies in the center of A, it must lie in B and B is a normal divisor of the group A. Since B is maximal, the centralizer of B in A coincides with B, and the factor group A/B is isomorphic to some subgroup of the group of automorphisms of B; and thus, A/B has a finite number of generators. Therefore, the arbitrary abelian subgroup $A/G^{(n)}$ of the group $H/G^{(n)}$ has a finite number of generators. Since the n-th commutator of $H/G^{(n)}$ reduces to 1, then $H/G^{(n)}$ is an A_5-group by the induction hypothesis, which proves the theorem.

§4. Periodic subgroups of A_4-groups

In view of Theorem 3, research on periodic subgroups of solvable A_1-groups reduces, on the one hand, to a study of subgroups of periodic solvable groups, and on the other hand, to research on periodic and, hence, finite subgroups of solvable A_4-groups. It is easy to see that neither the maximal periodic subgroups of a solvable A_4-group nor the Sylow subgroups will be conjugate in the general case. For example, the group with generators a, b, c and generating relations $a^4 = 1$, $a^{-1}ba = c$, $a^{-1}ca = b^{-1}$, $bc = cb$ has two classes of conjugate Sylow subgroups of order 4 and one class of Sylow subgroups of order 2. This situation is characteristic of the general case.

Theorem 9. *The periodic subgroups of a solvable A_4-group lie in a finite number of conjugate classes.*

The proof is based on a well-known lemma, the proof of which will be given here for completeness. Let us say that the subgroup G of an additively written abelian group A is a *definitely determined* subgroup if to every function $a(g)$ defined on G, with values in A there corresponds some element $\int a(g)\,dg$ in A, satisfying the conditions:

1) $\int [a(g) + b(g)]\,dg = \int a(g)\,dg + \int b(g)\,dg;$

2) $\int [a(g)]\,L\,dg = \left(\int a(g)\,dg\right) L$, where L is an automorphism of A;

3) $\int a(gxh)\,dx = \int a(x)\,dx;$

4) $\int a(x)\,dx = a$, if $a(x)$ is a.

Lemma 6. *Let the group G contain a solvable normal divisor R; let there be a chain of solvable normal divisors R of G satisfying the inclusion relations $R \supset R_1 \supset \cdots \supset R_s \supset 1$. Suppose that all factor groups R_i/R_{i+1} are definitely determined with respect to the group G/R. Then G has a complementary subgroup H with the properties $H \cap R = 1$; $HR = G$, and all the complementary subgroups having these properties are conjugate in G.*

First, we shall consider the case when the normal divisor R is abelian and is definitely determined with respect to G/R. Choose a representative g_α from each residue class $\alpha \in G/R$. As usual, set $g_\alpha g_\beta = g_{\alpha\beta} r(\alpha, \beta)$. From the relation $g_\alpha \cdot g_\beta g_\gamma = g_\alpha g_\beta \cdot g_\gamma$, we obtain

$$r(\alpha, \beta\gamma) + r(\beta, \gamma) = r(\alpha\beta, \gamma) + r(\alpha, \beta)\,A_\gamma, \tag{2}$$

where A_γ is the automorphism induced in R by the element g_γ. On setting

$$\int r(\alpha, \beta)\,d\alpha = -r(\beta),$$

in (2), we obtain

$$-r(\beta\gamma) + r(\beta, \gamma) = -r(\gamma) - r(\beta)\,A_\gamma. \tag{3}$$

If we replace $g_\alpha r(\alpha)$ by h_α, we obtain

$$h_\beta h_\gamma = g_\beta g_\gamma \cdot g_\gamma^{-1} r(\beta)\, g_\gamma \cdot r(\gamma) = h_{\beta\gamma} r^{-1}(\beta\gamma)\, r(\alpha, \gamma) \cdot r(\beta)\, A_\gamma \cdot r(\gamma),$$

i.e., in view of relation (3), $h_\beta h_\gamma = h_{\beta\gamma}$. This shows that the elements of the form h_α generate a subgroup H in G, such that the relations $H \cap R = 1$, $HR = G$ hold.

Now let us show that complementary subgroups are conjugate. Suppose the relations

$$G = HR = KR, \quad h = k_h r_h \quad (h \in H,\ k_h \in K,\ r_h \in R),$$

hold, where H and K are complementary subgroups. Since H and K are groups, the relations

$$ab = k_{ab} r_{ab} = k_a k_b \cdot r_a^{k_b} r_b \quad (a, b \in H); \text{ or } r_{ab} = r_a A_{k_b} + r_b.$$

must hold. Integrating with respect to a, we obtain $r = rA_{k_b} + r_b$, where r is

$\int r_a da$. From this it follows

$$rhr^{-1} = rk_h r_h r^{-1} = k_h (rA_{k_h} + r_h - r) = k_h, \text{ i.e., } rHr^{-1} = K.$$

The proof of the lemma in the case that the normal divisor R has a solvable chain $R \supset R_1 \supset \cdots \supset R_s \supset 1$ proceeds by induction on the length s of this chain. Indeed, let the lemma be proved for groups in which the length of the chain is $s-1$. The factor group G/R_s has a normal divisor R/R_s which has a chain of length $s-1$. Then, there is a complementary subgroup H_1/R_s in G/R_s. The group H_1 contains an abelian normal divisor R_s determined by means of H_1/R_s. By the case already examined, H_1 has a complementary subgroup H which is the desired complementary subgroup in G. In a similar manner, it is proved that the complementary subgroups are conjugate.

We have the following corollary:

Corollary. If the group G contains a complete nilpotent normal divisor R, if R has no elements of finite order, and if the factor group G/R is finite, then every finite subgroup of G is contained in a finite subgroup isomorphic to G/R; and all finite subgroups of G which are isomorphic to G/R are conjugate in G.

To prove this, note that the factor groups of the upper central series of R are complete abelian torsion-free groups. To define a mean of a function $a(a)$ defined on G/R with value in a complete abelian torsion-free group, we can take $(1/n)\sum_a a(a)$, where n is the number of elements in G/R. By Lemma 6, there must be a subgroup H of G which is isomorphic to G/R; and all subgroups isomorphic to G/R are conjugate in G. Let B be some finite subgroup of G. Then, to every element b in B corresponds an element h_b in H such that $b = h_b r_b$ ($r_b \in R$). The set of all h_b generates a subgroup H_1 of H. Since H_1 and B are finite subgroups of the group BR, isomorphic to BR/R, then as already proved, there is an r in R such that B is $r^{-1}H_1r$. Therefore, B is contained in the subgroup $r^{-1}Hr$, which is isomorphic to H.

The corollary is equivalent to the following assertion:

Let the group G contain a complete locally nilpotent torsion-free normal divisor R. Let the factor group G/R be locally finite and denumerable. Then, every locally finite subgroup of the group G is contained in some complementary subgroup H with the properties $H \cap R = 1$, $HR = G$. If the factor group G/R is finite, then all the complementary subgroups are conjugate in G.

Let $b_1, \cdots, b_n$ be elements of the group G, the images of which in G/R form a subgroup of G/R. Let B be the subgroup of G generated by the elements b_i. Let R_1 be the intersection $B \cap R$. Then, R_1 is a normal divisor with finite index in B and has a finite number of generators, since B does. Since the group R is locally nilpotent, R_1 must be a nilpotent normal divisor of B. Let R_1^* be the complement of R_1 in R. By [8], R_1^* is a normal divisor in the group BR since the index of R_1^* in BR_1^* is finite. The corollary of Lemma 7 shows that BR_1^* is a finite subgroup, the image of which in G/R is the same as the image of B. By the same corollary, it follows that all finite subgroups of G having the same image in G/R are conjugate in G. If G/R is finite, everything is proved. If G/R is infinite, it can be considered as the union of an ascending chain $\overline{G}_1 \subset \overline{G}_2 \subset \cdots$ of finite subgroups. It must be shown that there are finite subgroups G_k in G which have images in G/R which coincide with the subgroups $\overline{G}_k$. Since finite subgroups with the same images must be identical, the subgroups G_i may be chosen so that they form an ascending chain. The join of all the G_i gives the desired complementary subgroup in G.

Now, let us turn to the proof of Theorem 9.

A) Let the group G contain an abelian normal torsion-free divisor R of finite rank; and a finite subgroup H, isomorphic to the factor group G/R. In addition, we suppose that R contains no central elements of the group G. Let the group R be written additively. The elements of G can be represented as pairs (h, r), $h \in H$, $r \in R$, with multiplication table

$$(h_1, r_1)(h_2, r_2) = (h_1 h_2, r_1 A_{h_2} + r_2), \tag{4}$$

where A_{h_2} is the automorphism induced in R by the element h_2. Let us extend R to an abelian group R^* with division (i.e., rational coefficients). The automorphisms of R can be uniquely extended to automorphisms of R^*, and the group G can be considered as a subgroup of the group G^* of pairs (h, r), $h \in H$, $r \in R^*$, with multiplication table that of formula (4). Let $B = \{b_1, \cdots, b_s\}$ be a finite subgroup of G; set $b_i = (a_i, r_i)$, $a_i \in H$, $r_i \in R (i = 1, 2, \cdots, s)$. By the corollary of Lemma 6, B is the transform by some element x in R^* of some subgroup A generated by the elements $a_1, \cdots, a_s$, i.e.,

$$b_i = (1, x)\,(\overline{a_i}, 0)\,(1, -x) = (a_i, x(A_i - E)), \tag{5}$$

so that

$$x(A_i - E) \in R. \tag{6}$$

Let R_0^* be the set of all elements x in R^* which satisfy relation (6). Every finite subgroup of G which has image A in G/R can be obtained from A by transforming A by the elements of the set R_0^*. If r_1, r_2 are elements of R_0^* with difference in R, the subgroups $r_1^{-1} A r_1$, $r_2^{-1} A r_2$ can be transformed into one another by the element $r_1 - r_2$ in G. Thus, the number of inconjugate finite subgroups of the group G which have images in G/R equal to A is no greater than the number of cosets of R_0^* with respect to R.

Now, let us show that the number of these cosets is finite. The group R^* can be considered as a linear space over the field of rational numbers. With respect to a given coordinate system, we can represent a linear transformation A_i by a matrix. The correspondence $a_1 \to A_i$ is a representation of the finite group A. This representation decomposes over the field of complex numbers into irreducible classes, but the identical representation does not occur in view of the assumption that R contains no central elements of G. We note the relation

$$(A_i - E)(A_j - E) = (A_i A_j - E) - (A_i - E) - (A_j - E).$$

Thus, the set of all possible linear combinations of the differences $A_i - E$ generates a hypercomplex matrix system. This system decomposes into absolutely irreducible sets, and the null set is not among them. Thus, the set being considered contains non-singular matrices. Let $U = \Sigma\, m_i (A_i - E)$ be one of these matrices. We can suppose that the coefficients m_i are integers. Since the transformations

A_i are automorphisms of the group R, U is also an automorphism of this group. Since the determinant of U is not 0, the transformation U, considered as a transformation on the space R^*, has an inverse U^{-1}, and we can write

$$U^{-1} = \alpha_0 + \alpha_1 U + \cdots + \alpha_{n-1} U^{n-1},$$

with a_i certain rational numbers. Let N be the common denominator of these numbers, then we can write $NU^{-1} = k_0 + k_1 U + \cdots + k_{n-1} U^{n-1}$, where the k_i are integers.

Let x be an element of R_0^*. From (6), it follows it follows that $xU = r$ is in R. From this, we obtain $Nx = rNU^{-1} = k_0 r + \cdots + k_{n-1} rU^{n-1}$, i.e., $Nx \in R$. This shows that the difference group R_0^*/R is a periodic abelian group of finite rank, the elements of which have bounded order. Thus, the order of R_0^*/R is finite, as was to be shown.

We were supposing that the normal divisor R contains no central elements of the group G. It is easy to remove that restriction. Let Z be the set of central elements of G which are contained in R. The factor group G/Z contains an abelian torsion-free normal divisor R/Z of finite rank, and a finite subgroup AZ/Z isomorphic to G/R, such that the rank of R/Z is less than the rank of R. Thus, we can use induction on the rank of R; for a finite subgroup in G/Z breaks down into a finite number of conjugate subgroups. But if the subgroups A and B of the group G are equal mod Z, they must coincide. Thus, the number of classes of conjugate subgroups in G does not exceed the corresponding number of classes in G/Z.

B) Now, let us suppose that the group G contains a torsion-free abelian normal divisor R of finite rank and finite index in G. The group G can be thought of as an extension of the group R by means of the finite group $\{a_1, \cdots, a_s\} = G/R$ corresponding to a system $r(a_i, a_j)$ of factors and automorphisms A_i. Imbed R in a complete group R^* of the same rank. The automorphisms A_i can be uniquely extended to automorphisms of R^*. Let G^* be the extension of R^* by the group G/R with the same system of factors $r(a_i, a_j)$ and extended automorphisms A_i. By the corollary of Lemma 6, the group G^* can be thought of as a product AR, where A is a finite subgroup of G^*, isomorphic to G^*/R^*. The original group G can be thought of as a subgroup of the group G^*. If $a_1, \cdots, a_s$ are the elements of A, then for some r_i in R^*, the element $a_i r_i$ is in G.

Let N be a natural number for which the element Nr_i is in $R\,(i = 1, 2, \cdots, s)$, and let R_0^* be the set of elements x in R^* such that Nx is in R. Now, R_0^* is a normal divisor in G^*, and G coincides with AR_0^*; hence, the index of R in R_0^* and the index of G in AR_0^* are finite. By the A) part of the proof, the number of classes of finite conjugate subgroups in the group AR_0^* is finite. Since G has finite index in AR_0^*, it also follows that the number of classes of finite conjugate groups in the group G is also finite.

C) Now we consider the general case. Let G be a solvable A_4-group. By Theorem 3, G has a descending chain of normal divisors $G \supset G_1 \supset \cdots \supset G_s \supset 1$, such that every factor of the chain is either a finite group or else an abelian torsion-free group of finite rank. If s is 0, the conclusion of the theorem is obvious. Thus, we can apply induction on s, and assume that the factor group G/G_s has a finite number of classes of conjugate finite subgroups. Let the groups A_i/G_s be representatives of all these classes, so that a member of every class of finite subgroups of the group G/G_s is contained in one of the groups A_i/G_s. If the group G_s is finite, then every A_i is also finite, and everything is proved. If the group G_s is infinite, then by B) every one of the groups A_i contains a finite number of classes of finite conjugate subgroups, that is, the number of finite conjugate subgroups contained in G is finite.

The theorem is proved. From the theorem follows the assertion that *in every solvable group of type A_1, the maximal periodic subgroups split into a finite number of conjugate classes.*

Indeed, by Theorem 3 every solvable A_1-group G has a periodic normal divisor P with factor group of type A_4. This normal divisor is contained in every maximal periodic subgroup, and the number of classes of conjugate maximal periodic subgroups in G cannot exceed the number of classes of finite conjugate subgroups in G/P.

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ON DIRECT DECOMPOSITIONS OF GROUPS

L. Ya. KULIKOV

Introduction

One of the fundamental problems in the theory of direct decompositions of groups is to decide whether two given direct decompositions of a group possess isomorphic refinements.

Suppose that two direct decompositions of a group G are given[*]

$$G=\sum_{a\in M} D_a \tag{D}$$

and

$$G=\sum_{\nu\in N} A_\nu . \tag{A}$$

By Ω_A we denote the set of all normal idempotent endomorphisms of G corresponding[**] to the direct decomposition (A). With the direct decompositions (D) and (A) of G we associate a topological space $R(\mathrm{D}, \Omega_{\mathrm{A}})$ whose points are the direct summands D_a, $a \in M$, of (D), while the closed sets of $R(\mathrm{D}, \Omega_A)$ are defined as follows: a set of direct summands of (D) will be called a closed subset of $R(\mathrm{D}, \Omega_{\mathrm{A}})$ if the subgroup of G generated by this set of direct summands is admissible (characteristic) with respect to the set of endomorphisms Ω_{A}.

It turns out that the existence of isomorphic refinements for the two given direct decompositions (D) and (A) of G is closely connected with properties of the space $R(\mathrm{D}, \Omega_{\mathrm{A}})$ and the similarly constructed space $R(\mathrm{A}, \Omega_{\mathrm{D}})$.

The main problem studied in the present paper is that of the existence of isomorphic refinements for the two direct decompositions (D) and (A) of G if the space $R(\mathrm{D}, \Omega_{\mathrm{A}})$ corresponding to these decompositions is a Kolmogorov space.[***]

We shall prove that if G is a countable group and $R(\mathrm{D}, \Omega_{\mathrm{A}})$ a Kolmogorov space, then (D) and (A) possess centrally isomorphic refinements. The same result holds for a non-countable group G, provided at least one of the following conditions is satisfied: 1) the number of direct summands in (D) is finite; 2) every direct summand in (A) has not more than countable cardinal; 3) every descending sequence of closed subsets of $R(D, \Omega_{\mathrm{A}})$ breaks off after a finite number of steps (see §8). Furthermore, we give conditions under which a fixed direct decomposition of a group and an arbitrary other decomposition of the same group have centrally isomorphic refinements (see §8, 9) and also conditions for the existence of centrally isomorphic refinements of any two direct decompositions of a group (see §10).

An abelian group is called *completely decomposable* if it is a direct sum of subgroups of rank *1*.[****]

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[*] Throughout the paper we shall use the *additive* notation for the group operation both in commutative and non-commutative groups.

[**] See §3.

[***] See §1.

[****] We recall that an abelian group is said to be of rank *1* if an arbitrary finite subset of elements generates a cyclic subgroup.

The results obtained in §8, 9, 10 are applied to the solution of the following two problems.

Do any two direct decompositions of a completely decomposable group possess isomorphic refinements?

Is every direct summand of a completely decomposable group itself completely decomposable?

These two problems are closely connected: a solution of one of them leads to a solution of the other. For torsion-free abelian groups of finite rank and for a class of torsion free abelian groups of infinite rank the two problems have been solved by Baer [6]. In §12 we shall study the solution of these problems for *countable* abelian groups, for periodic groups, and for a class of non-countabe mixed groups.

The results of the first ten sections of the paper and their proofs remain valid for groups with an arbitrary operator domain.

In conclusion we mention that the method of this paper can be used to solve the problem of the existence of directly similar refinements for two given direct decompositions of the unit element of a complete modular lattice.

§1

Let

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

be a direct decomposition of a group G and Ω an arbitrary set of endomorphisms of G. With (D) and Ω we associate a topological space $R(\mathrm{D}, \Omega)$ whose points are the direct summands D_α, $\alpha \in M$, of (D), while the closed sets of $R(\mathrm{D}, \Omega)$ are defined as follows: a set of direct summands of (D) will be called a closed subset of $R(\mathrm{D}, \Omega)$ if the subgroup of G generated by this set of direct summands is admissible with respect to Ω.

(1.1) Definition. A topological space is called a *Kolmogorov space** if any two distinct one-point subsets of the space have distinct closures.

This definition is equivalent to the following:

(1.2) Definition. A topological space is called a *Kolmogorov space* if of any two distinct one-point subsets of the space at least one is separated from the other by a closed set.

(1.3) Definition. A direct decomposition (D) of a group G not containing null summands is said to be *partially ordered* with respect to a set of endomorphisms Ω if $R(\mathrm{D}, \Omega)$ is a Kolmogorov space.

It is well-known that the intersection of an arbitrary set of Ω-subgroups** and the group generated by the union of an arbitrary collection of Ω-subgroups of G are also Ω-subgroups of G. Therefore, if (D) is partially ordered with respect to Ω, then $R(\mathrm{D}, \Omega)$ is a discrete topological space.

By the symbol $C(\mathrm{D})$ we shall denote the set of all those subgroups of G

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* Translator's note: or T_0-space.

** A subgroup of G that is admissible with respect to a set of endomorphisms Ω will be called an Ω-subgroup of G.

which are generated by an arbitrary collection of direct summands of (D). By the symbol $\Delta(\mathrm{D}, \Omega)$ we shall denote the set of all Ω-subgroups belonging to $C(\mathrm{D})$.

By the symbol $V(D_\alpha, \mathrm{D}, \Omega)$ we shall denote the least group of $\Delta(\mathrm{D}, \Omega)$ that contains the direct summand D_α of (D). When Ω is fixed, we shall often denote $V(D_\alpha, \mathrm{D}, \Omega)$ by $\overline{D}_\alpha$.

From the definition of the space $R(\mathrm{D}, \Omega)$ it follows that the set $\{D_\gamma\}_{\gamma \in \Gamma}$ of direct summands of (D) is closed in $R(\mathrm{D}, \Omega)$ if and only if the subgroup

$$\sum_{\gamma \in \Gamma} D_\gamma$$

of G is an element of $\Delta(\mathrm{D}, \Omega)$. Moreover, it is not difficult to see that if the set $\{D_\gamma\}_{\gamma \in V}$ is the closure in $R(\mathrm{D}, \Omega)$ of the one-point set $\{D_\alpha\}$, then $\overline{D}_\alpha = \sum_{\gamma \in V} D_\gamma$. Therefore, two distinct one-point sets $\{D_\alpha\}$ and $\{D_\beta\}$ of $R(\mathrm{D}, \Omega)$ have distinct closures if and only if $\overline{D}_\alpha \neq \overline{D}_\beta$. Thus, taking the definitions (1.1), (1.2), (1.3) into account it is easy to see that the following proposition holds.

(1.4) *In order that the direct decomposition* (D) *of a group* G *be partially ordered with respect to a set of endomorphisms* Ω *it is necessary and sufficient that one of the following two conditions is satisfied:*

a) *The decomposition* (D) *contains no null summands, and the following relation holds*

$$\overline{D}_\alpha \neq \overline{D}_\beta \quad (\alpha \neq \beta,\ \alpha,\ \beta \in M).$$

b) *For any two distinct direct summands* D_α *and* D_β *of* (D) *there exists a group in the set* $\Delta(\mathrm{D}, \Omega)$ *containing one of the direct summands but not the other.*

By Q we denote the set of all those Ω-subgroups of $V(D_\alpha, \mathrm{D}, \Omega)$ which belong to $C(\mathrm{D})$ and have only the null element as their intersection with D_α. It is easy to see that the subgroup of G generated by an arbitrary collection of subgroups of Q also belongs to Q. Therefore, there exists a largest subgroup among the groups belonging to Q, that is, a group belonging to Q and containing every group of Q as a subgroup. We shall denote this largest group of Q by $U(D_\alpha, \mathrm{D}, \Omega)$. Thus, the symbol $U(D_\alpha, \mathrm{D}, \Omega)$ denotes the largest Ω-subgroup of $V(D_\alpha, \mathrm{D}, \Omega)$ that belongs to $C(\mathrm{D})$ and has only the null element as intersection with D_α. When Ω is fixed, we shall often denote the group $U(D_\alpha, \mathrm{D}, \Omega)$ by $\check{D}_\alpha$.

Obviously $\check{D}_\alpha \neq \overline{D}_\alpha$, if D_α is not the null group.

(1.5) *If the direct decomposition* (D) *is partially ordered with respect to the set of endomorphisms* Ω, *then the relation*

$$\overline{D}_\beta \subset \check{D}_\alpha \quad (\alpha,\ \beta \in M) \tag{a}$$

holds if and only if

$$\overline{D}_\beta \subset \overline{D}_\alpha \text{ and } \beta \neq \alpha \quad (\alpha,\ \beta \in M). \tag{b}$$

Proof. 1. Let us assume that (D) is partially ordered with respect to Ω. Then $\check{D}_\alpha \neq \{0\}$, $\overline{D}_\alpha \neq \check{D}_\alpha$, and therefore (a) implies (b).

2. Let us assume now that the groups D_α and D_β satisfy the relation (6). By (1.4)

$$\overline{D}_\beta \neq \overline{D}_\alpha \quad (\beta \neq \alpha,\ \alpha,\ \beta \in M). \tag{1}$$

As $\overline{D}_\beta \subset C(\mathrm{D})$, we have either

$$\overline{D}_\beta \cap D_\alpha = \{0\} \tag{2}$$

or

$$D_\alpha \subset \overline{D}_\beta. \tag{3}$$

But (3) is impossible for $\alpha \neq \beta$. For if $D_\alpha \subset \overline{D}_\beta$, then $\overline{D}_\alpha \subset \overline{D}_\beta$, and this in conjunction with (b) gives $\overline{D}_\beta = \overline{D}_\alpha$, in contradiction to (1). Therefore, we have (2). On the basis of (b) and (2) we deduce that $\overline{D}_\beta$ is a subgroup of $\overline{D}_\alpha$ having only the null element as intersection with D_α. Moreover, $\overline{D}_\beta \in \Delta(\mathrm{D}, \Omega)$. Therefore, $\overline{D}_\beta \subset \check{D}_\alpha$. Thus, (b) implies (a).

(1.6) *If the direct decomposition* (D) *of a group G is partially ordered with respect to a set of endomorphisms Ω, then the following direct decomposition holds:*

$$\overline{D}_\alpha = D_\alpha + \check{D}_\alpha \quad (\alpha \in M). \tag{a}$$

Proof. Let us assume that (D) is partially ordered with respect to Ω. Let D_α be an arbitrary direct summand of (D). As $D_\alpha \in C(\mathrm{D})$, there exists a subset V of M such that

$$\overline{D}_\alpha = \sum_{\beta \in V} D_\beta. \tag{1}$$

The group

$$\sum_{\beta \in V - \{\alpha\}} D_\beta$$

is, obviously, the largest subgroup of D_α belonging to $C(\mathrm{D})$ and not containing D_α. But $\check{D}_\alpha$ is the largest Ω-subgroup belonging to $C(\mathrm{D})$ and having only the null element as intersection with D_α. Therefore,

$$\check{D}_\alpha \subset \sum_{\beta \in V - \{\alpha\}} D_\beta. \tag{2}$$

Further, as $D_\beta \subset \overline{D}_\alpha$ for $\beta \in V$

$$\overline{D}_\beta \subset \overline{D}_\alpha \quad (\beta \in V). \tag{3}$$

Since (D) is partially ordered with respect to Ω, we have by (1.4)

$$\overline{D}_\beta \neq \overline{D}_\alpha \quad (\beta \neq \alpha,\ \beta \in V). \tag{4}$$

From (3) and (4) it follows by (1.5) that

$$\overline{D}_\beta \subset \check{D}_\alpha \quad (\beta \neq \alpha,\ \beta \in V),$$

hence

$$D_\beta \subset \check{D}_\alpha \quad (\beta \neq \alpha,\ \beta \in V). \tag{5}$$

Comparing (2) and (5) we obtain

$$\check{D}_\alpha = \sum_{\beta \in V - \{\alpha\}} D_\beta. \tag{6}$$

Now (1) and (6) show that the direct decomposition (a) holds.

Let

$$G = \sum_{i \in M} D_i \tag{D}$$

be an arbitrary direct decomposition of a group G and let Ω be a set of endomorphisms of G. Two direct summands D_i and D_k of (1) will be called Ω-equivalent if $V(D_i, \mathrm{D}, \Omega) = V(D_k, \mathrm{D}, \Omega)$. It is not difficult to see that the set of all direct summands of (D) splits into disjoint classes of Ω-equivalent summands. The decomposition (D) is partially ordered with respect to Ω if and only if every class of Ω-equivalent summands contain only one direct summand of (D), that is, if any two distinct direct summands of (D) are not Ω-equivalent. If (D) is not partially ordered with respect to Ω and if the number of Ω-equivalent classes of direct summands of (D) is not less than two, then on combining all direct summands of (D) that belong to one and the same class into a single summand we obtain a new direct decomposition of G which is now partially ordered wuth respect to Ω.

§2.

(2.1) Definition. A set L is said to be *partially ordered* if for some pairs α, β of distinct elements of L an order relation denoted by $\alpha < \beta$ is established satisfying the following conditions:

1. The relations $\alpha < \beta$ and $\beta < \alpha$ exclude each other.
2. If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.

If α and β are two distinct elements of a partially ordered set L, then we shall say that β is incomparable with α if neither $\alpha < \beta$ nor $\beta < \alpha$ holds.

In what follows we shall not exclude from our consideration a partially ordered set in which any two distinct elements are incomparable.

(2.2) Definition. A subset Γ of a partially ordered set L is called *closed* if from $\alpha \in \Gamma$, $\lambda < \alpha$ it follows that $\lambda \in \Gamma$.

It is easy to see that the following proposition holds:

(2.3) *The sum and the intersection of an arbitraty set of closed subsets of a partially ordered set are closed subsets.*

(2.4) Definition. Two partially ordered sets are called *similar* if an order preserving one-to-one correspondence can be set up between them.

Let us suppose that the direct decomposition (D) is partially ordered with respect to the set of endomorphisms Ω. The discrete space $R(\mathrm{D},\Omega)$ can be turned in a natural manner into a partially ordered set if we put for any two distinct summands D_α and D_β of (D) that $D_\beta < D_\alpha$ if and only if D_β belongs to the closure of the one-point set $\{D_\alpha\}$ or, what is the same, if $D_\beta \subset \overline{D}_\alpha$. It is not difficult to see that the order relation so introduced satisfies the conditions 1 and 2 of the definition (2.1) and, therefore, turns $\{D_\alpha\}_{\alpha \in M}$ into a partially ordered set which we shall denote by the symbol $\theta(\mathrm{D}, \Omega)$.

If (D) is a direct decomposition of G, partially ordered with respect to Ω, then we can introduce an order relation in the index set M in the following way: if α and β are two distinct elements of M, then we put $\beta < \alpha$ if and only if $D_\beta < D_\alpha$ or, what is the same, if $D_\beta \subset \overline{D}_\alpha$. The order relation so introduced turns the index set M into a partially ordered set which we shall denote by the symbol $L(\mathrm{D}, \Omega)$. It is easy to see that the partially ordered sets $\theta(\mathrm{D}, \Omega)$ and $L(\mathrm{D}, \Omega)$ are similar and that the correspondence $D_\alpha \leftrightarrow \alpha$, $\alpha \in M$ is a similarity relation.

(2.5) *Let*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

be a direct decomposition of G, *partially ordered with respect to a set of endomorphisms* Ω. *A subgroup*

$$\sum_{\gamma \in \Gamma} D_\gamma ,$$

where $\Gamma \subset M$, *is admissible with respect to* Ω *if and only if the index set* Γ *is a closed subset of the partially ordered set* $L(\mathrm{D}, \Omega)$.

Proof. 1. We put

$$H = \sum_{\gamma \in \Gamma} D_\gamma . \tag{1}$$

We shall assume that Γ is a closed subset of the partially ordered set $L(\mathrm{D}, \Omega)$ and shall show that then H is admissible with respect to Ω. We prove first that

$$D_\alpha \subset H \quad (\alpha \in \Gamma). \tag{2}$$

Suppose that $\alpha \in \Gamma$ and that D_β, $\beta \in M$ is a direct summand of (D) and a subgroup of $\overline{D}_\alpha$. Then $\beta \leq \alpha$. Hence, it follows, since $\alpha \in \Gamma$ and Γ is closed in $L(\mathrm{D}, \Omega)$, that $\beta \in \Gamma$ and, in fact, by (1)

$$D_\beta \subset H \quad (D_\beta \subset \overline{D}_\alpha,\ \alpha \in \Gamma). \tag{3}$$

As $\overline{D}_\alpha$ is generated by some subset of direct summands of (D) we deduce from (3) that (2) holds. Moreover, since $D_\alpha \subset \overline{D}_\alpha$, we have from (1) and (2)

$$H = [\bigcup_{\alpha \in \Gamma} \overline{D}_\alpha].^* \tag{4}$$

Now taking into account that $\overline{D}_\alpha$ is an Ω-subgroup of G for every $\alpha \in M$ we deduce from (4) that H is an Ω-subgroup of G.

2. We shall assume now that

$$H = \sum_{\gamma \in \Gamma} D_\gamma$$

is an Ω-subgroup of G and shall show that then the set Γ is closed in $L(\mathrm{D}, \Omega)$. Let $\alpha \in \Gamma$. We have to prove that an arbitrary element $\beta \in L(\mathrm{D}, \Omega)$, with $\beta < \alpha$, belongs to Γ. As $\beta < \alpha$, we have $D_\beta \subset D_\alpha$ and, in fact

$$D_\beta \subset D_\alpha. \tag{5}$$

Since $\alpha \in \Gamma$, we have $D_\alpha \subset H$. Moreover, H is an Ω-subgroup of G belonging to the set $C(\mathrm{D})$. Further, $\overline{D}_\alpha$ is the least Ω-subgroup of G belonging to $C(\mathrm{D})$ and containing D_α. Therefore,

$$\overline{D}_\alpha \subset H. \tag{6}$$

From (5) and (6) we obtain $D_\beta \subset H$, that is,

$$D_\beta \subset \sum_{\gamma \in \Gamma} D_\gamma .$$

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[*] If $A, B, C, \cdots$ are subsets of a group G, the symbol $[A, B, C, \cdots]$ denotes the smallest subgroup of G containing $A, B, C, \cdots$.

Hence, $\beta \in \Gamma$. This shows that Γ is a closed subset of $L(\mathrm{D}, \Omega)$.

(2.6) **Definition.** Let λ be an element of a partially ordered set L. The symbol $V(L, \lambda)$ shall denote the set of all elements $a \in L$ with $a \leq \lambda$. The set $V(L, \lambda)$ is called the *combinatorial closure* of λ in the partially ordered set L.

The symbol $U(L, \lambda)$ shall denote the set of all elements $a \in L$ with $a < \lambda$.

It is very easy to see that the following propositions hold:

(2.7) *If* λ *is an element of a partially ordered set* L, *then the sets* $V(L, \lambda)$ *and* $U(L, \lambda)$ *are closed in* L.

(2.8) *If* Γ *is a closed subset of a partially ordered set* L *and* $\lambda \in \Gamma$, *then the subsets* $V(L, \lambda)$ *and* $U(L, \lambda)$ *belong to* Γ.

(2.9) *Let*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

be a direct decomposition of a group G, *partially ordered with respect to a set of endomorphisms* Ω. *We denote by* V_α *the combinatorial closure of the element* α *in the partially ordered set* $L(\mathrm{D}, \Omega)$, $\alpha \in M$, *and put* $U_\alpha = V_\alpha - \{\alpha\}$. *Then the following equations hold:*

$$\overline{D}_\alpha = \sum_{\beta \in V_\alpha} D_\beta \quad (\alpha \in M), \tag{a}$$

$$\check{D}_\alpha = \sum_{\beta \in U_\alpha} D_\beta \quad (\alpha \in M). \tag{b}$$

Proof. Let $\alpha \in M$. By definition $\overline{D}_\alpha$ is the largest Ω-subgroup of G belonging to $C(\mathrm{D})$ and containing $\overline{D}_\alpha$. Since $\overline{D}_\alpha \in C(\mathrm{D})$ there exists a subset Γ of M such that

$$\overline{D}_\alpha = \sum_{\beta \in \Gamma} D_\beta\,. \tag{1}$$

Since $\overline{D}_\alpha$ is an Ω-admissible subgroup of G, the set Γ is closed in $L(\mathrm{D}, \Omega)$ by (2.5). Hence, it follows from (2.8), as $\alpha \in \Gamma$, that Γ contains the combinatorial closure V_α of α,

$$V_\alpha \subset \Gamma. \tag{2}$$

By (2.7) V_α is a closed subset of $L(\mathrm{D}, \Omega)$. According to (2.5)

$$\sum_{\gamma \in V_\alpha} D_\gamma$$

is, therefore, an admissible subgroup with respect to Ω. Moreover,

$$\overline{D}_\alpha \subset \sum_{\gamma \in V_\alpha} D_\gamma\,,$$

because $\alpha \in V_\alpha$. But $\overline{D}_\alpha$ is the least Ω-subgroup of G, belonging to $C(\mathrm{D})$ and containing D_α. Therefore,

$$\overline{D}_\alpha \subset \sum_{\gamma \in V_\alpha} D_\gamma\,. \tag{3}$$

Now we deduce from (1), (2), and (3) that (a) holds.

From the definition of $\check{D}_\alpha$ it follows that $\check{D}_\alpha \in C(D)$. Further, by (1.6) $\overline{D}_\alpha = D_\alpha + \check{D}_\alpha$. The equation (a) now follows from this and (b). This proves the proposition (2.9).

Let F be a subset of a partially ordered set L. An element $a \in F$ is called a minimal (maximal) element of F, if there are no elements λ in F with $\lambda < a (\lambda > a)$.

We shall say that a subset F of partially ordered set L satisfies the *descending chain condition* if every descending sequence

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots$$

of elements of F contains only a finite number of terms.

We shall say that a subset F of a partially ordered set L satisfies the *minimal condition* if every non-empty subset of F has at least one minimal element.

It is easy to see that a subset F of a partially ordered set L satisfies the minimal condition if and only if it satisfies the descending chain condition.

§3

In this section we shall expound certain definitions and auxiliary propositions which will be required in the following sections.

Let

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

be a direct decomposition of a group G. With every direct summand A_ν of (A) we associate an endomorphism ϕ_ν of G mapping every element $g \in G$ into its component in the direct summand A_ν of (A). The endomorphism ϕ_ν will be called *the endomorphism corresponding to the direct summand* A_ν *of* (A).

It is easy to see that ϕ_ν is an independent normal* endomorphism of G. The set $\{\phi_\nu\}_{\nu \in N}$ will be called *the set of endomorphisms corresponding to the decomposition* (A) and will be denoted by Ω_A.

(3.1) Let

$$G = \sum_{\nu \in \Gamma} F_\nu \tag{F}$$

be a direct decomposition of G *and* ψ_ν *the endomorphism corresponding to the direct summand* F_ν *of* (F). *A subgroup* H *of* G *is admissible with respect to the endomorphism* ψ_ν *if and only if* H *satisfies the following condition:*

$$\psi_\nu H = F_\nu \cap H. \tag{a}$$

Proof. From (a) if follows, obviously, that $\psi_\nu H \subset H$ and, therefore, condition (a) is sufficient. Let us show that it is also necessary. Let H be a subgroup of G, admissible with respect to ψ_ν, so that

$$\psi_\nu H \subset H. \tag{1}$$

From the definition of ψ_ν it follows that

$$\psi_\nu x = x \quad (x \in F_\nu),$$

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*An endomorphism of a group is said to be *normal* if it is permutable with all inner automorphisms of the group.

so that

$$F_\nu \cap H \subset \psi_\nu H. \tag{2}$$

On the other hand, since $\psi_\nu H \subset F_\nu$, it follows from (1) that

$$\psi_\nu H \subset H \cap F_\nu. \tag{3}$$

Comparing (2) and (3) we obtain (a). Thus, the condition (a) is also necessary.

(3.2) *Let*

$$G = \sum_{\nu \in \Gamma} F_\nu \tag{F}$$

be a direct decomposition of a group G. *In order that a normal subgroup* H *of* G *be admissible with respect to the set of endomorphisms* Ω_F *it is necessary and sufficient that there is the direct decomposition*

$$H = \sum_{\nu \in \Gamma} (F_\nu \cap H). \tag{H}$$

Proof. 1. Let us prove the necessity of the condition. By ψ_ν we denote the endomorphism corresponding to the direct summand F_ν of (F) so that

$$\Omega_F = \{\psi_\nu\}_{\nu \in \Gamma}.$$

Let H be a normal subgroup of G, admissible with respect to Ω_F, that is

$$\psi_\nu H \subset H \quad (\nu \in \Gamma). \tag{1}$$

We show that the decomposition (H) holds. Since $\psi_\nu H$ is the component of the normal subgroup H in the direct summand F_ν of (F), it is easy to see that $\psi_\nu H$ is a normal subgroup of H, $\nu \in \Gamma$, and that we have $[\bigcup_{\nu \in \Gamma} \psi_\nu H] = \sum_{\nu \in \Gamma} \psi_\nu H$. Hence,

$$H \subset \sum_{\nu \in \Gamma} \psi_\nu H. \tag{2}$$

From (1) and (2) we conclude that

$$H = \sum_{\nu \in \Gamma} \psi_\nu H. \tag{3}$$

Further, from (1) it follows by (3.1) that

$$\psi_\nu H = F_\nu \cap H \quad (\nu \in \Gamma). \tag{4}$$

Comparing (3) and (4) we obtain (H).

2. Let us prove the sufficiency of the condition. Suppose that H is a normal subgroup of G for which the decomposition (H) holds. Let $\psi_a \in \Omega_F$. Since ψ_a is an endomorphism of G it follows from (H) that

$$\psi_a H = [\bigcup_{\nu \in \Gamma} \psi_a (F_\nu \cap H)] \quad (\psi_a \in \Omega_F). \tag{5}$$

As ψ_a is the endomorphism of G corresponding to the direct summands F_a of (F), we have the following relation:

$$\psi_a F_\nu = \{0\} \quad (a \neq \nu,\ \nu \in \Gamma), \tag{6}$$

$$\psi_a x = x \quad (x \in F_a). \tag{7}$$

From (6) follows

$$\psi_a(H \cap F_\nu) = \{0\} \quad (a \neq \nu,\ \nu \in \Gamma). \tag{8}$$

Further, by (7)

$$\psi_a(H \cap F_a) = H \cap F_a. \tag{9}$$

Comparing (5), (8), (9) we obtain $\psi_a H = H \cap F_a$, that is,

$$\psi_a H \subset H \quad (\psi_a \in \Omega_{\mathrm{F}}). \tag{10}$$

The relation (10) shows that H is an admissible subgroup with respect to Ω_{F}.

(3.3) *If*

$$G = \sum_{a \in M} D_a \tag{D}$$

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

are two decompositions of G *and* $H \in \Delta(\mathrm{D}, \Omega_{\mathrm{A}})$, *then the following direct decomposition holds:*

$$H = \sum_{\nu \in N} (A_\nu \cap H).$$

Proof. This proposition is a consequence of (3.2), since by assumption $H \in \Delta(\mathrm{D}, \Omega_{\mathrm{A}})$ and hence, H is a normal subgroup of G admissible with respect to Ω_{A}.

(3.4) *If*

$$G = \sum_{a \in M} D_a \tag{D}$$

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

are two direct decompositions of G, $\overline{D}_a$ *the least group of* $\Delta(\mathrm{D}, \Omega_{\mathrm{A}})$ *containing* D_a, *and* $\check{D}_a$ *the largest subgroup of* $\overline{D}_a$ *belonging to* $\Delta(\mathrm{D}, \Omega_{\mathrm{A}})$ *and not containing* D_a, *then the following direct decompositions hold:*

$$\overline{D}_a = \sum_{\nu \in N} (A_\nu \cap \overline{D}_a) \quad (a \in M), \tag{a}$$

$$\check{D}_a = \sum_{\nu \in N} (A_\nu \cap \check{D}_a) \quad (a \in M). \tag{b}$$

This proposition is a particular case of the preceding.

(3.5) **Definition.** A direct decomposition of G

$$G = \sum_{a \in M} D_a \tag{D}$$

is called *consistent* with the decomposition

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

if it is partially ordered with respect to the set of endomorphisms Ω_{A} corresponding to (A).

(3.6) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of G, *consistent with* (A). *By* H_P *we denote the direct sum* $\sum_{\lambda \in P} E_\lambda$, *where* $P \subset L$. *The direct decomposition*

$$H_P = \sum_{\nu \in N} (A_\nu \cap H_P)$$

holds if and only if P *is a closed subset of the partially ordered set* $L(\mathrm{E}, \Omega_{\mathrm{A}})$.

This proposition follows immediately from (2.5) and (3.2).

(3.7) *Let* L *be a partially ordered set and let*

$$G = \sum_{\lambda \in L} E_\lambda, \tag{E}$$

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

be two direct decompositions of G, *where* (E) *does not contain any direct summands null. If for an arbitrary closed subset* P *of* L *we have*

$$H_P = \sum_{\nu \in N} (A_\nu \cap H_P) \tag{a}$$

where $H_P = \sum_{\lambda \in P} E_\lambda$, *then* (E) *is consistent with* (A).

Proof. Let E_i and E_k be two distinct direct summands of (E). We show that there exists an Ω_{A}-subgroup of G containing one of these direct summands, but not the other. This will prove in virtue of (1.4) that (E) is consistent with (A). Suppose, for example, that $i < k$ or that i is incomparable with k. Then, obviously, k is not contained in the combinatorial closure V_i of i in the partially ordered set L,

$$k \notin V_i. \tag{1}$$

V_i is, evidently, a closed subset of L.

Putting $P = V_i$ in (a) we obtain

$$H_{V_i} = \sum_{\nu \in N} (A_\nu \cap H_{V_i}). \tag{2}$$

In view of (3.2) the relation (2) shows that H_{V_i} is an admissible subgroup with respect to Ω_{A}. Moreover, H_{V_i} contains E_i and, by (1) does not contain E_k, because by assumption E_k is not the null subgroup. Thus, (E) is consistent with (A).

(3.8) *Let*

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

be a direct decomposition of G, ψ_ν *the endomorphism of* G *corresponding to the direct summand* A_ν *of* (A), *and* $\{B_\lambda\}_{\lambda \in Q}$ *the set of subgroups of* G *that are admissible with respect to* ψ_ν. *Then the following relation holds:*

$$A_\nu \cap [\bigcup_{\lambda \in Q} B_\lambda] = [\bigcup_{\lambda \in Q} (A_\nu \cap B_\lambda)]. \tag{a}$$

Proof. As ψ_ν is an endomorphism of G, we have

$$\psi_\nu[\bigcup_{\lambda\in Q} B_\lambda]=[\bigcup_{\lambda\in Q}\psi_\lambda B_\lambda]. \tag{1}$$

Further, the groups B_λ are by assumption admissible with respect to ψ_ν, that is,

$$\psi_\nu B_\lambda \subset B_\lambda \quad (\lambda \in Q), \tag{2}$$

hence, by (1)

$$\psi_\nu[\bigcup_{\lambda\in Q} B_\lambda]\subset[\bigcup_{\lambda\in Q} B_\lambda]. \tag{3}$$

From (2) and (3) it follows by (3.1) that

$$\psi_\nu B_\lambda = A_\nu \cap B_\lambda \quad (\lambda \in Q) \tag{4}$$

$$\psi_\nu[\bigcup_{\lambda\in Q} B_\lambda]=A_\nu \cap [\bigcup_{\lambda\in Q} B_\lambda]. \tag{5}$$

Comparing (1) and (5) we obtain

$$A_\nu \cap [\bigcup_{\lambda\in Q} B_\lambda]=[\bigcup_{\lambda\in Q}\psi_\nu B_\lambda];$$

and hence, (a) follows by (4).

(3.9) *Let*

$$G = \sum_{\nu\in N} A_\nu$$

be a direct decomposition of G *and* $\{B_\lambda\}_{\lambda\in Q}$ *the set of subgroups of* G *that are admissible with respect to* Ω_Λ. *Then*

$$A_\nu \cap [\bigcup_{\lambda\in Q} B_\lambda] = [\bigcup_{\lambda\in Q}(A_\nu\cap B_\lambda)].$$

This proposition follows immediately from the preceding.

(3.10) *If* A, B, D, *are subgroups of a group* G, C *a subgroup of* A, *and if*

$$B = C + D, \tag{1}$$

then

$$A \cap B = C + A \cap D. \tag{2}$$

Proof. Let x be an arbitrary element of $A \cap B$. By (1) $x = c + d$, where $c \in C$, $d \in D$; therefore, $d = (x - c) \in A$, $d \in A \cap D$. Thus, x is the sum of the elements c and d belonging to C and $A \cap D$, respectively. Therefore, $A \cap B$ is generated by C and $A \cap D$. Moreover, taking (1) into account, it is easy to see that C and $A \cap D$ are normal subgroups of $A \cap B$ whose intersection consists of the null element only. Therefore, (2) holds.

§4

(4.1) **Theorem.** *In order that a direct decomposition of a group* G *have a common refinement with any other direct decomposition of* G *it is necessary and sufficient that every direct summand of this decomposition is admissible with respect to the set of all normal idempotent endomorphisms of* G.

Proof. 1. We shall first prove the sufficiency of the condition.

Let

$$G = \sum_{\alpha \in M} H_\alpha \tag{H}$$

be a direct decomposition of a group G in which every direct summand is admissible with respect to the set Ω of all normal idempotent endomorphisms of G, and let

$$G = \sum_{k \in \Gamma} F_k \tag{F}$$

be any other direct decomposition of G. As H_α is admissible with respect to Ω, it is also admissible with respect to the set Ω_F of normal idempotent endomorphisms corresponding to (F). Therefore, by (3.2), we have

$$H_\alpha = \sum_{k \in \Gamma} (F_k \cap H_\alpha) \quad (\alpha \in M). \tag{1}$$

Replacing in (H) every direct summand H_α by the formula (1) we obtain the direct decomposition

$$G = \sum_{\substack{k \in \Gamma \\ \alpha \in M}} (F_k \cap H_\alpha),$$

which is, evidently, a common refinement of (H) and (F). Thus, the condition of the theorem is sufficient.

2. We shall now prove the necessity of the condition. Let

$$G = \sum_{i \in N} H_i \tag{2}$$

be a direct decomposition of G having a common refinement with any other direct decomposition of G. We shall show that every direct summand H_i of (2) is admissible with respect to the set Ω of all normal idempotent endomorphisms of G. Let $\psi \in \Omega$, e the identical automorphism of G and $\overline{\psi} = e - \psi$. We put $F_1 = \psi G$, $F_2 = \psi G$. It is well known that we have then

$$G = F_1 + F_2, \tag{3}$$

where ψ and $\overline{\psi}$ are, obviously, the idempotent endomorphisms of G corresponding to (3).

The decompositions (2) and (3) have a common refinement, because (2) has a common refinement with any other decomposition of G. Let

$$G = \sum_{n \in Q} C_n \tag{4}$$

be a common refinement of (2) and (3). By Z_{ki} we denote the direct sum of all those summands C_n in (4) which are contained in $F_k \cap H_i$. As (4) is a refinement of (2) and (3), we have then the direct decomposition

$$H = Z_{1i} + Z_{2i} \quad (i \in N). \tag{5}$$

Moreover, from the definition of Z_{ki} it follows that

$$Z_{1i} \subset F_1 \cap H_i,\ Z_{2i} \subset F_2 \cap H_i \quad (i \in N).$$

Hence, taking (3) into account, we obtain

$$Z_{1i} + Z_{2i} \subset F_1 \cap H_i + F_2 \cap H_i \quad (i \in N). \tag{6}$$

From (5) and (6) we deduce that

$$H_i = F_1 \cap H_i + F_2 \cap H_i \quad (i \in N). \tag{7}$$

Now, as ψ is the normal idempotent endomorphism corresponding to the direct summand F_1 of (3), it follows by (3.2) from (7) that every summand H_i of (2) is admissible with respect to ψ. This proves that every direct summand of (2) is admissible with respect to Ω. Thus, the condition of the theorem is also necessary, and the proof is complete.

Kuroš [3] and Fitting [9] have proved the following two theorems:

1. *A direct decomposition of a group* G

$$G = \sum_{\alpha \in M} A_\alpha$$

has a common refinement with any other direct decomposition of G *if and only if all summands* A_α *are characteristic (admissible) with respect to the set of normal automorphisms of* G.

2. *A direct summand of* G *that is admissible (characteristic) with respect to the set of all normal automorphisms of* G *is also admissible with respect to the set of all normal endomorphisms of* G.

The following proposition is a corollary of these two theorems and of Theorem (4.1).

(4.2) *In order that a direct summand of a group be admissible (characteristic) with respect to the set of all normal endomorphisms or the set of all normal automorphisms of the group it is necessary and sufficient that it is admissible with respect to the set of all normal idempotent endomorphisms of the group.*

From (4.2) we deduce the following proposition:

(4.3) *A direct decomposition of a group is partially ordered with respect to the set of all normal endomorphisms or the set of all normal automorphisms of the group if and only if it is partially ordered with respect to the set of all normal idempotent endomorphisms of the group.*

§5

(5.1) *Let*

$$H = B + S \tag{a}$$

$$H = \sum_{\nu \in \Gamma} C_\nu + S, \tag{b}$$

be two direct decompositions of a group H. *We put*

$$B_\nu = B \cap (C_\nu + S) \quad (\nu \in \Gamma). \tag{c}$$

Then the following relations hold:

$$B_\nu + S = C_\nu + S \quad (\nu \in \Gamma), \tag{d}$$

$$B = \sum_{\nu \in \Gamma} B_\nu. \tag{e}$$

Proof. From (a) and (b) we deduce that

$$C_\nu + S \subset B + S \quad (\nu \in \Gamma),$$

hence, by (3.10) we have

$$C_\nu + S = B \cap (C_\nu + S) + S$$

or by (c) $C_\nu + S = B_\nu + S$, that is, (d) holds.

By (c)

$$[\bigcup_{\nu \in \Gamma} B_\nu] \subset B, \tag{1}$$

hence and from (a) we have

$$[\bigcup_{\nu \in \Gamma} B_\nu] \cap S = \{0\}. \tag{2}$$

From (a) and (b) we deduce that

$$B_\nu \cap [\bigcup_{\mu \in \Gamma - \{\nu\}} B_\mu] \subset (C_\nu + S) \cap (\sum_{\mu \in \Gamma - \{\nu\}} C_\mu + S) = S.$$

Hence, by (2) we have

$$B_\nu \cap [\bigcup_{\mu \in \Gamma - \{\nu\}} B_\mu] = \{0\}. \tag{3}$$

Further, by (b) and (d)

$$H = [\bigcup_{\nu \in \Gamma} B_\nu, S]. \tag{4}$$

From (c) which defines B_ν it follows that B_ν is a normal subgroup of H, $\nu \in \Gamma$. Therefore, from (2), (3), and (4) we deduce the following direct decomposition:

$$H = \sum_{\nu \in \Gamma} B_\nu + S. \tag{5}$$

Finally, (e) follows from the relations (a), (1), and (5).

(5.2) *Let*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

be a direct decomposition of a group G, *consistent with the decomposition*

$$G = \sum_{\nu \in N} A_\nu. \tag{A}$$

We introduce the following notations:

$\overline{D}_\alpha$ *is the least group in* $\Delta(\mathrm{D}, \mathbf{\Omega}_\mathbf{A})$ *containing* D_α;

$\check{D}_\alpha$ *is the largest subgroup of* $\overline{D}_\alpha$ *belonging to* $\Delta(\mathrm{D}, \mathbf{\Omega}_\mathbf{A})$ *and not containing* D_α;

$E_{(\alpha, \nu)}$ *is the component of* $A_\nu \cap \overline{D}_\alpha$ *in the direct summand* D_α *of* (D).

Then the following relations hold:

$$\overline{D}_\alpha = D_\alpha + \check{D}_\alpha \quad (\alpha \in M), \tag{a}$$

$$\overline{D}_\alpha = \sum_{\nu \in N} (A_\nu \cap \overline{D}_\alpha) \quad (\alpha \in M), \tag{b}$$

$$\check{D}_\alpha = \sum_{\nu \in N} (A_\nu \cap \check{D}_\alpha) \quad (\alpha \in M), \tag{c}$$

$$A_\nu \cap \overline{D}_\alpha = C_{(\alpha,\nu)} + A_\nu \cap \check{D}_\alpha \quad (\nu \in N,\ \alpha \in M), \tag{d}$$

where
$$C_{(\alpha,\nu)} = A_\nu \cap (D_\alpha + \sum_{\gamma \in N - \{\nu\}} (A_\gamma \cap \check{D}_\alpha)),$$

$$E_{(\alpha,\nu)} = D_\alpha \cap [A_\nu \cap \overline{D}_\alpha, \check{D}_\alpha], \tag{e}$$

$$E_{(\alpha,\nu)} = D_\alpha \cap (C_{(\alpha,\nu)} + \check{D}_\alpha), \tag{f}$$

$$E_{(\alpha,\nu)} + \check{D}_\alpha = C_{(\alpha,\nu)} + \check{D}_\alpha \tag{g}$$

$$D_\alpha = \sum_{\nu \in N_\alpha} E_{(\alpha,\nu)},$$

where
$$N_\alpha = \mathop{\mathrm{E}}_{\nu} (\nu \in N,\ E_{(\alpha,\nu)} \neq [0]), \tag{h}$$

$$[A_\nu \cap \overline{D}_\alpha, \check{D}_\alpha] = E_{(\alpha,\nu)} + \check{D}_\alpha. \tag{i}$$

Proof. By assumption (D) is consistent with (A). Therefore, (a), (b), and (c) hold. For (a) follows from (1.6), (b) and (c) from (3.4).

From (a) and (c) follows

$$\overline{D}_\alpha = D_\alpha + \sum_{\gamma \in N} (A_\gamma \cap \check{D}_\alpha),$$

which can be written in the form

$$\overline{D}_\alpha = A_\nu \cap \check{D}_\alpha + D_\alpha + \sum_{\gamma \in N - \{\nu\}} (A_\gamma \cap \check{D}_\alpha). \tag{1}$$

Now (d) follows from (1) by (3.10).

Note that $A_\nu \cap \overline{D}_\alpha \subset \overline{D}_\alpha$ and that $\check{D}_\alpha$ is the direct sum of a certain set of direct summands of (D). Therefore, taking the definition of $E_{(\alpha,\nu)}$ into account, we can assert that $E_{(\alpha,\nu)}$ is also a component of $A_\nu \cap \overline{D}_\alpha$ in the direct summand D_α of the decomposition $\overline{D}_\alpha = D_\alpha + \check{D}_\alpha$ so that, in fact, (e) holds.

From (d), (a), and (b) we deduce that

$$[A_\nu \cap \overline{D}_\alpha, \check{D}_\alpha] = C_{(\alpha,\nu)} + \check{D}_\alpha. \tag{2}$$

Comparing (e) and (2), we obtain (f).

Replacing every direct summand $A_\nu \cap \overline{D}_\alpha$ in (b) by its expression (d) we obtain

$$\overline{D}_\alpha = \sum_{\nu \in N} C_{(\alpha,\nu)} + \sum_{\nu \in N} (A_\nu \cap \check{D}_\alpha). \tag{3}$$

We now apply proposition (5.1) and replace in it H, B, S, C_ν, Γ by $\overline{D}_\alpha, D_\alpha, \check{D}_\alpha, C_{(\alpha,\nu)}, N$, respectively. According to (5.1) from (a), (3), and (f) follows (g) and

$$\overline{D}_\alpha = \sum_{\nu \in N} E_{(\alpha,\nu)}. \tag{4}$$

If we now exclude in (4) the direct summands that are null groups, we obtain (h).

Finally, comparing (2) and (g) we obtain (c).

(5.3) *If the decomposition*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

is consistent with the decomposition

$$G = \sum_{\nu \in N} A_\nu, \tag{A}$$

then the component of $A_\nu \cap \overline{D}_\alpha$ *in the direct summand* D_α *of* (D) *is the null group if and only if* $A_\nu \cap \overline{D}_\alpha \subset \check{D}_\alpha$.

Proof. We assume that (D) is consistent with (A). Then, by (5.2i) the component $E_{(\alpha,\nu)}$ of $A_\nu \cap \overline{D}_\alpha$ in the direct summand D_α of (D) satisfies the relation

$$[A_\nu \cap \overline{D}_\alpha, \check{D}_\alpha] = E_{(\alpha,\nu)} + \check{D}_\alpha.$$

This shows that $E_{(\alpha,\nu)}$ is the null group if and only if $A_\nu \cap \overline{D}_\alpha \subset \check{D}_\alpha$.

(5.4) **Definition.** Let A be a subgroup of G,

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

a direct decomposition of G and Ω and arbitrary set of endomorphisms of G. The symbol $S(A, \mathrm{E}, \Omega)$ shall denote the set

$$\underset{\lambda}{\mathrm{E}} \, (A \cap \overline{E}_\lambda \not\subset \check{E}_\lambda, \lambda \in L),$$

where $\overline{E}_\lambda$ is the least subgroup of the set $\Delta(\mathrm{E}, \Omega)$ containing E_λ, and $\check{E}_\lambda$ the largest subgroup of $\overline{E}_\lambda$ belonging to $\Delta(\mathrm{E}, \Omega)$ and not containing E_λ.

(5.5) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of G, *consistent with the decomposition*

$$G = \sum_{\nu \in N} A_\nu. \tag{A}$$

Then

$$L = \bigcup_{\nu \in N} S(A_\nu, \mathrm{E}, \Omega_A). \tag{a}$$

Proof. Let us assume that (a) does not hold. Then there exists an index $\lambda \in L$, satisfying the relation

$$A_\nu \cap \overline{E}_\lambda \subset \check{E}_\lambda \quad (\nu \in N). \tag{1}$$

By hypothesis (E) is consistent with (A) so that by (3.4)

$$\overline{E}_\lambda = \sum_{\nu \in N} (A_\nu \cap \overline{E}_\lambda), \tag{2}$$

$$\check{E}_\lambda = \sum_{\nu \in N} (A_\nu \cap \check{E}_\lambda). \tag{3}$$

Comparing (1), (2), and (3) we obtain

$$\overline{E}_\lambda = \check{E}_\lambda$$

which is impossible, because $E_\lambda \subset \overline{E}_\lambda$, but $\check{E}_\lambda$ does not contain E_λ. This proves that (a) holds.

(5.6) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of G, *consistent with the decomposition*

$$G = \sum_{\nu \in N} A_\nu. \tag{A}$$

Then the following two properties of (E) *are equivalent:*

(a) *For every* $\lambda \in L$ *there exists not more than one direct summand* A_ν *in* (A) *for which the component of* $A_\nu \cap \overline{E}_\lambda$ *in the direct summand* E_λ *of* (E) *is not the null group.*

(b) *For every* $\lambda \in L$ *there exists not more than one direct summand* A_ν *in* (A), *satisfying the relation*

$$A_\nu \cap \overline{E}_\lambda \not\subset \check{E}_\lambda,$$

so that, in other words, for $\nu \neq \gamma$, ν, $\gamma \in N$, *the sets* $S(A_\nu, \mathrm{E}, \Omega_\Lambda)$ *and* $S(A_\gamma, \mathrm{E}, \Omega_\Lambda)$ *do not intersect.*

This proposition follows immediately from (5.3).

(5.7) **Definition.** Let

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

be two direct decompositions of a group G. The decomposition (E) is said to be *fully consistent* with (A) if it is consistent with (A) and has one of the properties (a) or (b) of (5.6).

(5.8) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, *fully consistent with the decomposition*

$$G = \sum_{\nu \in N} A_\nu. \tag{A}$$

Then the following relations hold:

$$L = \bigcup S(A_\nu, \mathrm{E}, \Omega_\Lambda), \tag{a}$$

$$S(A_\nu, \mathrm{E}, \Omega_\Lambda) \cap S(A_\gamma, \mathrm{E}, \Omega_\Lambda) = \emptyset \quad (\nu \neq \gamma, \nu, \gamma \in N), \tag{b}$$

where $\emptyset$ *is the empty set. Thus, there exists a single-valued mapping of* L *onto* N, *mapping the element* $\lambda \in L$ *onto the unique element* $\nu \in N$ *for which the relation*

$$\lambda \in S(A_\nu, \mathrm{E}, \Omega_\mathrm{A})$$

holds. This mapping will be denoted by the symbol ρ.

Proof. By hypothesis (E) is fully consistent with (A). Therefore, the relations (a) and (b) hold. For (a) follows from (5.5), (b) from the definition (5.7). The existence of the mapping ρ now follows from (a) and (b).

(5.9) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, *fully consistent with the decomposition*

$$G = \sum_{\nu \in N} A_\nu . \tag{A}$$

Then the following relations hold:

$$A_{\rho(\lambda)} \cap \overline{E}_\lambda \not\subset \check{E}_\lambda \quad (\lambda \in L), \tag{a}$$

$$A_\nu \cap \overline{E}_\lambda \subset \check{E}_\lambda \quad (\nu \neq \rho(\lambda),\ \nu \in N,\ \lambda \in L), \tag{b}$$

$$A_{\rho(\lambda)} \cap \overline{E}_\lambda \neq A_{\rho(\lambda)} \cap \check{E}_\lambda \quad (\lambda \in L), \tag{c}$$

$$A_\nu \cap \overline{E}_\lambda = A_\nu \cap \check{E}_\lambda \quad (\nu \neq \rho(\lambda),\ \nu \in N,\ \lambda \in L). \tag{d}$$

Proof. According to the definition of the mapping ρ

$$\lambda \in S(A_{\rho(\lambda)}, \mathrm{E}, \Omega_\mathrm{A}) \quad (\lambda \in L), \tag{1}$$

$$\lambda \notin S(A_\nu, \mathrm{E}, \Omega_\mathrm{A}) \quad (\nu \neq \rho(\lambda),\ \nu \in N,\ \lambda \in L). \tag{2}$$

Taking the definition (5.4) into account it is easy to see that the relations (a), (b) follow from (1), (2). Obviously, (c) follows from (a). Finally, by (b)

$$A_\nu \cap \overline{E}_\lambda \subset A_\nu \cap \check{E}_\lambda \quad (\nu \neq \rho(\lambda),\ \nu \in N,\ \lambda \in L),$$

and (d) follows because $\check{E}_\lambda \subset \overline{E}_\lambda$.

(5.10) Theorem. *Let*

$$G = \sum_{a \in M} D_a \tag{D}$$

be a direct decomposition of a group G, *consistent with the decomposition*

$$G = \sum_{\nu \in N} A_\nu . \tag{A}$$

By L *we denote the set of all ordered pairs* (a, ν), $a \in M$, $\nu \in N$, *for which the component* $E_{(a,\nu)}$ *of* $A_\nu \cap \overline{D}_a$ *in the direct summand* D_a *of* (D) *is not the null group. Then the following direct decomposition holds.*

$$G = \sum_{\lambda \in L} E_\lambda , \tag{E}$$

and this is a refinement of (D) *and fully consistent with* (A).

Proof. 1. It is easy to see that the decomposition (E) holds. For by (5.2h)

$$D_\alpha = \sum_{\lambda\in L^{(\alpha)}} E_\lambda, \tag{1}$$

where $L^{(\alpha)}$ is the set of those pairs $(\gamma, \nu) \in L$ in which the first index γ is fixed and equal to α. From the definition of L it follows that

$$L = \bigcup_{\alpha\in M} L^{(\alpha)}. \tag{2}$$

If we now replace in (D) every direct summand D_α by its expression (1) as direct sum of the groups E_λ and take (2) into account, then we obtain the direct decomposition (E) which, evidently, is a refinement of (D).

2. We shall show now that (E) is consistent with (A). It is not difficult to see that

$$\Delta(\mathrm{D}, \Omega_{\mathrm{A}}) \subset \Delta(\mathrm{E}, \Omega_{\mathrm{A}}). \tag{3}$$

This relation holds because (E) is a refinement of (D).

Let $E_{(\alpha,\nu)}$ and $E_{(\alpha_1,\nu_1)}$ be two distinct direct summands of (E). We shall show that there is a group in the set $\Delta(\mathrm{E}, \Omega_{\mathrm{A}})$ containing one of these direct summands, but not the other.

First case. $\bar{D}_{\alpha_1} \subset \check{D}_\alpha$. In this case, since $E_{(\alpha_1,\nu_1)} \subset D_{\alpha_1} \subset \bar{D}_{\alpha_1}$, we have

$$E_{(\alpha_1,\nu_1)} \subset \check{D}_\alpha. \tag{4}$$

Taking into account that $E_{(\alpha,\nu)}$ is a non-null subgroup of D_α and that $\bar{D}_\alpha = D_\alpha + \check{D}_\alpha$ we conclude that

$$E_{(\alpha,\nu)} \not\subset \check{D}_\alpha. \tag{5}$$

Further, since $\check{D}_\alpha \in \Delta(\mathrm{D}, \Omega_{\mathrm{A}})$ we have, by (3),

$$\check{D}_\alpha \in \Delta(\mathrm{E}, \Omega_{\mathrm{A}}). \tag{6}$$

The relations (4), (5), and (6) show that $\check{D}_\alpha$ is the required group containing $E_{(\alpha_1,\nu_1)}$, but not $E_{(\alpha,\nu)}$.

Second case. $\bar{D}_\alpha \subset \check{D}_{\alpha_1}$. Just as in the first case we find that $\check{D}_{\alpha_1}$ is the required group.

Third case. $\bar{D}_{\alpha_1} \not\subset \check{D}_\alpha$ and $\bar{D}_\alpha \not\subset \check{D}_{\alpha_1}$. In this case we have, obviously,

$$D_{\alpha_1} \cap \bar{D}_\alpha = \{0\} \quad \text{for } \alpha_1 \neq \alpha,$$

and as $E_{(\alpha_1,\nu_1)} \subset D_{\alpha_1}$ and $E_{(\alpha,\nu)} \subset D_\alpha$, it follows that

$$E_{(\alpha_1,\nu_1)} \cap (E_{(\alpha,\nu)} + \check{D}_\alpha) = \{0\}. \tag{7}$$

Making use of (1) we shall show that (7) also holds for $\alpha_1 = \alpha$. The groups $E_{(\alpha,\nu)}$ and $E_{(\alpha_1,\nu_1)}$, $\nu \neq \nu_1$, are distinct direct summands of (1). If we denote by B the sum of the direct summands in (1), other than $E_{(\alpha,\nu)}$ and $E_{(\alpha_1,\nu_1)}$, then we obtain therefore,

$$D_\alpha = E_{(\alpha_1,\, \nu_1)} + E_{(\alpha,\, \nu)} + B. \tag{8}$$

Since $\overline{D}_\alpha = D_\alpha + \check{D}_\alpha$, (8) implies the equation

$$\overline{D}_\alpha = E_{(\alpha_1,\, \nu_1)} + (E_{(\alpha,\, \nu)} + \check{D}_\alpha + B,$$

which shows that (7) also holds for $\alpha_1 = \alpha$. As $E_{(\alpha,\, \nu)}$ is not the null group, we deduce from (7) that

$$E_{(\alpha_1,\, \nu_1)} \not\subset (E_{(\alpha,\, \nu)} + \check{D}_\alpha). \tag{9}$$

Let us show that

$$(E_{(\alpha,\, \nu)} + \check{D}_\alpha) \in \Delta(\mathrm{E}, \Omega_{\mathrm{A}}). \tag{10}$$

The group $A_\nu \cap \overline{D}_\alpha$ is admissible with respect to Ω_{A} because it is a subgroup of the direct summand A_ν of (A). The group $\check{D}_\alpha$ is also admissible with respect to Ω_{A} because $\check{D}_\alpha \in \Delta(\mathrm{D}, \Omega_{\mathrm{A}})$. Moreover, by (5.2i)

$$E_{(\alpha,\, \nu)} + \check{D}_\alpha = [A_\nu \cap \overline{D}_\alpha, \check{D}_\alpha].$$

Therefore, the group $(E_{(\alpha,\, \nu)} + \check{D}_\alpha)$ is also admissible with respect to Ω_{A}.

$(E_{(\alpha,\, \nu)} + \check{D}_\alpha)$ is the direct sum of a certain set of direct summands E_λ of (E). This follows from the fact that $\check{D}_\alpha$ is the direct sum of a certain set of direct summands of (D) and that every direct summand of (D) is, by (1), a direct sum of summands E_λ of (E). Therefore, the relation (10) holds.

Now (9) and (10) show that $E_{(\alpha,\, \nu)} + \check{D}_\alpha$ is the required group in $\Delta(\mathrm{E}, \Omega_{\mathrm{A}})$ containing $E_{(\alpha,\nu)}$, but not $E_{(\alpha_1,\, \nu_1)}$.

Thus, we have proved that (E) is consistent with (A).

3. Let us show now that (E) is fully consistent with (A).

According to the definition of $E_{(\alpha,\, \nu)}$

$$E_{(\alpha,\nu)} \subset D_\alpha \quad (\alpha \in M,\ \nu \in N). \tag{11}$$

$\overline{E}_{(\alpha,\, \nu)}$ is the least group belonging to $\Delta(\mathrm{E}, \Omega_{\mathrm{A}})$ and containing $E_{(\alpha,\, \nu)}$. But $\overline{D}_\alpha$ also belongs to $\Delta(\mathrm{E}, \Omega_{\mathrm{A}})$ and, by (11) contains $E_{(\alpha,\, \nu)}$. Therefore,

$$\overline{E}_{(\alpha,\, \nu)} \subset \overline{D}_\alpha;$$

hence,

$$A_\beta \cap E_{(\alpha,\, \nu)} \subset A_\beta \cap \overline{D}_\alpha. \tag{12}$$

Let $E_{(\alpha,\, \nu)} = E_\lambda$ be an arbitrary direct summand of (E). By $C_{\lambda,\beta}$ we denote the component of $A_\beta \cap \overline{E}_{(\alpha,\, \nu)}$, $\beta \in N$, in the direct summand $E_{(\alpha,\, \nu)}$ of (E). By $E_{(\alpha,\, \beta)}$ we have denoted previously the conponent of $A_\beta \cap \overline{D}_\alpha$ in the direct summand D_α of (D). Taking into account that (E) is a refinement of (D) we deduce from (11) and (12) that

$$C_{\lambda,\, \beta} \subset E_{(\alpha,\, \beta)} \quad (\lambda \in L,\ \alpha \in M,\ \beta \in N). \tag{13}$$

It is easy to see that

$$E_{(\alpha,\nu)} \cap E_{(\alpha,\beta)} = \{0\} \quad (\alpha \in M,\ \beta \neq \nu,\ \beta \in N). \tag{14}$$

For if $(\alpha, \beta) \notin L$, then according to the definition of L, $E_{(\alpha,\beta)}$ is the null group. But if $(\alpha, \beta) \in L$, and $\beta \neq \alpha$, then (14) also holds because $E_{(\alpha,\nu)}$ and $E_{(\alpha,\beta)}$ are in this case distince direct summands of (E). Bearing in mind that $C_{\lambda,\beta} \subset E_{(\alpha,\nu)}$ we deduce from (13) and (14) that

$$C_{\lambda,\beta} = \{0\} \quad (\lambda \in L,\ \beta \in N,\ \beta \neq \nu). \tag{15}$$

The relation (15) shows that the component of $A_\beta \cap \overline{E}_{(\alpha,\nu)}$ in the direct summand $E_{(\alpha,\nu)}$ of (E), $(\alpha, \nu) \in L$, is the null group for every $\beta \in N$, other than ν. This proves that (E) is fully consistent with (D).

(5.11) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, *fully consistent with the decomposition*

$$G = \sum_{\nu \in N} A_\nu . \tag{A}$$

We introduce the following notations:

C_λ *is the component of* E_λ *in the direct summand* $A_{\rho(\lambda)}$ of (A);

$\overline{E}_\lambda$ *is the least group in* $\Delta(\mathrm{E}, \Omega_\Lambda)$ *containing* E_λ;

$\check{E}_\lambda$ *is the largest subgroup of* $\overline{E}_\lambda$ *belonging to* $\Delta(\mathrm{E}, \Omega_\Lambda)$ *and not containing* E_λ;

$H_P = \sum_{\lambda \in P} E_\lambda$.

Then the following relations hold:

$$\overline{E}_\lambda = E_\lambda + \check{E}_\lambda \quad (\lambda \in L), \tag{a}$$

$$H = \sum_{\nu \in N} (A_\nu \cap H) \quad (H \in \Delta(\mathrm{E}, \Omega_\Lambda)), \tag{b}$$

$$H_P = \sum_{\nu \in N} (A_\nu \cap H_P) \quad (P \in Z), \tag{c}$$

$$\overline{E}_\lambda = \sum_{\nu \in N} (A_\nu \cap \overline{E}_\lambda) \quad (\lambda \in L), \tag{d}$$

$$\check{E}_\lambda = \sum_{\nu \in N} (A_\nu \cap \check{E}_\lambda) \quad (\lambda \in L), \tag{e}$$

$$\overline{E}_\lambda = [A_{\rho(\lambda)} \cap \overline{E}_\lambda, \check{E}_\lambda] \quad (\lambda \in L), \tag{f}$$

$$C_\lambda = A_{\rho(\lambda)} \cap [E_\lambda, \sum_{\nu \in N - \{\rho(\lambda)\}} (A_\nu)] \quad (\lambda \in L), \tag{g}$$

$$C_\lambda = A_{\rho(\lambda)} \cap (E_\lambda + \sum_{\nu \in N - \{\rho(\lambda)\}} (A_\nu \cap \check{E}_\lambda)) \quad (\lambda \in L), \tag{h}$$

$$A_{\rho(\lambda)} \cap \overline{E}_\lambda = C_\lambda + A_{\rho(\lambda)} \cap \check{E}_\lambda \quad (\lambda \in L), \tag{i}$$

$$C_\lambda + \check{E}_\lambda = E_\lambda + \check{E}_\lambda \quad (\lambda \in L), \tag{j}$$

$$[A_{\rho(\lambda)} \cap \bar{E}_\lambda, \check{E}_\lambda] = C_\lambda + \check{E}_\lambda \quad (\lambda \in L). \tag{k}$$

Proof. By hypothesis, (E) is fully consistent with (A). Therefore, (a)–(e) hold. For (a) and (b) hold by (1.6) and (3.3); (c) follows from (b) by (2.5); (d) and (e) are special cases of (b), because

$$\bar{E}_\lambda, \check{E}_\lambda \in \Delta(\mathrm{E}, \Omega_A).$$

By (d) we have

$$\bar{E}_\lambda = A_{\rho(\lambda)} \cap \bar{E}_\lambda + \sum_{\lambda \in N-\{\rho(\lambda)\}} (A_\nu \cap \bar{E}_\lambda). \tag{1}$$

Moreover, as (E) is consistent with (A), we have by (5.9)

$$A_\nu \cap \bar{E}_\lambda = A_\nu \cap \check{E}_\lambda \quad (\nu \neq \rho(\lambda), \nu \in N), \tag{2}$$

hence,

$$\sum_{\nu \in N-\{\rho(\lambda)\}} (A_\nu \cap \bar{E}_\lambda) \subset \check{E}_\lambda. \tag{3}$$

Now (f) follows from (1) and (3), because $\check{E}_\lambda \subset \bar{E}_\lambda$.

Since C_λ is the component of E_λ in the direct summand $A_{\rho(\lambda)}$ of (A), we have (g). Furthermore, taking into account that $E_\lambda \subset \bar{E}_\lambda$ and $A_\nu \cap \bar{E}_\nu \subset A_\nu$, $\nu \in N$, we conclude from (A) and (d) that the component of E_λ in the direct summand $A_{\rho(\lambda)} \cap \bar{E}_\lambda$ of (d) is C_λ, that is,

$$C_\lambda = (A_{\rho(\lambda)} \cap \bar{E}_\lambda) \cap (E_\lambda + \sum_{\nu \in N-\{\rho(\lambda)\}} (A_\nu \cap \bar{E}_\lambda)),$$

and as $E_\lambda \subset \bar{E}_\lambda$

$$C_\lambda = A_{\rho(\lambda)} \cap (E_\lambda + \sum_{\nu \in N-\{\rho(\lambda)\}} (A_\nu \cap \bar{E}_\nu)),$$

which in conjunction with (2) gives (h).

From (a) and (e) follows

$$\bar{E}_\lambda = E_\lambda + \sum_{\nu \in N-\{\rho(\lambda)} (A_\lambda \cap \check{E}_\lambda)$$

or

$$\bar{E}_\lambda = A_{\rho(\lambda)} \cap \check{E}_\lambda + E_\lambda + \sum_{\nu \in N-\{\rho(\lambda)\}} (A_\nu \cap \check{E}_\lambda),$$

hence, by (3.10) we obtain

$$A_{\rho(\lambda)} \cap \bar{E}_\lambda = A_{\rho(\lambda)} \cap \check{E}_\lambda + A_{\rho(\lambda)} \cap (E_\lambda + \sum_{\nu \in N-\{\rho(\lambda)\}} (A_\nu \cap \check{E}_\lambda)). \tag{4}$$

Now (i) follows from (h) and (4).

Further, from (f) and (i) we deduce (j). Finally, comparing (a), (f), and (j) we obtain (k).

(5.12) *Let* (D), (A), *and* (E) *be the direct decompositions of a group* G *dis-*

cussed in Theorem (5.10). *If the minimal condition holds for the subset* $S(A_\nu, \mathrm{D}, \Omega_\mathrm{A})$ *of the partially ordered set* $L(\mathrm{D}, \Omega_\mathrm{A})$, *then it also holds for the subset* $S(A_\nu, \mathrm{E}, \Omega_\mathrm{A})$ *of the partially ordered set* $L(\mathrm{E}, \Omega_\mathrm{A})$. *Furthermore, if the minimal condition holds for* $L(\mathrm{D}, \Omega_\mathrm{A})$, *then it also holds for* $L(\mathrm{E}, \Omega_\mathrm{A})$.

Proof. 1. Let (α, i) and (α, k) be two distinct elements of L, $i \neq k$. We shall show that (α, i) and (α, k), regarded as elements of $L(\mathrm{E}, \Omega_\mathrm{A})$ are incomparable.

As $E_{(\alpha, i)}$ and $E_{(\alpha, k)}$ are two distinct direct summands of (E), we have

$$E_{(\alpha, i)} \cap E_{(\alpha, k)} = \{0\}. \tag{1}$$

Moreover, since $\check{E}_{(\alpha, k)} \subset D_\alpha$,

$$E_{(\alpha, i)} \cap \check{E}_{(\alpha, k)} = \{0\}. \tag{2}$$

As $\overline{E}_{(\alpha, k)} = E_{(\alpha, k)} + \check{E}_{(\alpha, k)}$, by (5.11a), we conclude from (1) and (2) that

$$E_{(\alpha, i)} \cap \overline{E}_{(\alpha, k)} = \{0\} \quad (i \neq k). \tag{3}$$

Similarly, we see that

$$E_{(\alpha, k)} \cap \overline{E}_{(\alpha, i)} = \{0\} \quad (i \neq k). \tag{4}$$

(3) and (4) show that the elements (α, i) and (α, k) of the partially ordered set $L(\mathrm{E}, \Omega_\mathrm{A})$ are incomparable.

2. We shall now show that a relation

$$(\alpha, i) < (\beta, k) \quad ((\alpha, i), (\beta, k) \in L(\mathrm{E}, \Omega_\mathrm{A})) \tag{5}$$

implies that

$$\alpha < \beta \quad (\alpha, \beta \in L(\mathrm{D}, \Omega_\mathrm{A})). \tag{6}$$

By (5)

$$E_{(\alpha, i)} \subset \overline{E}_{(\beta, k)}. \tag{7}$$

As $\overline{E}_{(\beta, k)} \subset \overline{D}_\beta$, from (7) follows

$$E_{(\alpha, i)} \subset \overline{D}_\beta. \tag{8}$$

Since $E_{(\alpha, i)} \subset D_\alpha$, we deduce from (8) that

$$D_\alpha \subset \overline{D}_\beta,$$

hence,

$$\alpha \leq \beta \quad (\alpha, \beta \in L(\mathrm{D}, \Omega_\mathrm{A})). \tag{9}$$

If $\alpha \neq \beta$, then, by what we have proved under 1, the elements (α, i) and (β, k) are incomparable. But we have assumed that (5) holds. Therefore,

$$\alpha \neq \beta. \tag{10}$$

Comparing (9) and (10) we now obtain (6).

3. On the basis of 2 it is easy to see that the following assertion holds:

(a) if β is a minimal element of a subset $F \subset L(\mathrm{D}, \Omega_\mathrm{A})$ and $(\beta, k) \in L$, then (β, k) is a minimal element of the subset $F^* = \underset{(\alpha, \nu)}{\mathrm{E}} ((\alpha, \nu) \in L, \alpha \in F)$ of the partially ordered set $L(\mathrm{E}, \Omega_{\mathrm{A}})$. Thus, if the minimal condition holds in F, then it also holds in F^*.

4. From (a) we deduce that if the minimal condition holds in $L(\mathrm{D}, \Omega_{\mathrm{A}})$, then it also holds in $L(\mathrm{E}, \Omega_{\mathrm{A}})$.

5. We shall now prove the following statement:

(b) a relation

$$(a, i) \in S(A_\nu, \mathrm{E}, \Omega_{\mathrm{A}}) \tag{11}$$

implies that

$$a \in S(A_\nu, \mathrm{D}, \Omega_{\mathrm{A}}). \tag{12}$$

For if (12) does not hold, then

$$A_\nu \cap \overline{D}_a \subset \check{D}_a$$

and so

$$A_\nu \cap \overline{E}_{(a, i)} \cap \overline{D}_a \subset \overline{E}_{(a, i)} \cap \check{D}_a.$$

As

$$E_{(a, i)} \subset \overline{D}_a \text{ and } \overline{E}_{(a, i)} \cap \check{D}_a \subset \check{E}_{(a, i)},$$

we obtain, hence,

$$A_\nu \cap \overline{E}_{(a, i)} \subset \check{E}_{(a, i)},$$

which is impossible if (11) holds.

6. Now we can deduce from (a) and (b) that if the minimal condition holds for the subset $S(A_\nu, \mathrm{D}, \Omega_{\mathrm{A}})$ of $L(\mathrm{D}, \Omega_{\mathrm{A}})$, then it also holds for the subset $S(A_\nu, \mathrm{E}, \Omega_{\mathrm{A}})$ of $L(\mathrm{E}, \Omega_{\mathrm{A}})$.

This completes the proof of proposition (5.12).

(5.13) *Let*

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

be a decomposition of a group G *into the direct sum of indecomposable subgroups and*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

a direct decomposition of G, *fully consistent with* (A). *Then every direct summand* E_λ *in* (E) *can be replaced by the direct summand* $A_{\rho(\lambda)}$ *of* (A), *and* (A) *and* (E) *are, therefore, centrally isomorphic.*[*]

Proof. By C_λ, $\lambda \in L$, we denote the component of E_λ in the direct summand $A_{\rho(\lambda)}$ of (A). By (5.11j)

$$C_\lambda + \check{E}_\lambda = E_\lambda + \check{E}_\lambda \quad (\lambda \in L). \tag{1}$$

No summand E_λ of (E) is the null group because (E) is consistent with (A). Hence, it follows from (1) that C_λ is not the null group of G. From (1) and (E) we deduce that C_λ is a direct summand of G. As C_λ is a subgroup of $A_{\rho(\lambda)}$, it follows that C_λ is a non-null direct summand of $A_{\rho(\lambda)}$. But by hypothesis every summand of (A) is indecomposable. Therefore,

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[*] See §8 for the definition.

$$C_\lambda = A_{\rho(\lambda)} \quad (\lambda \in L). \tag{2}$$

From (1) and (2) we conclude that in (E) every summand E_λ can be replaced by $A_{\rho(\lambda)}$ of (A), that is,

$$G = A_{\rho(\lambda)} + \sum_{\lambda' \in L-\{\lambda\}} E_{\lambda'} = E_\lambda + \sum_{\lambda' \in L-\{\lambda\}} E_{\lambda'} \qquad (\lambda \in L),$$

so that for every $\lambda \in L$ the groups E_λ and $A_{\rho(\lambda)}$ are centrally isomorphic and, hence, (E) and (A) are centrally isomorphic.

(5. 14) Theorem. *Let*

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

be a decomposition of a group G into the direct sum of indecomposable subgroups and

$$G = \sum_{a \in M} D_a \tag{D}$$

a direct decomposition of G consistent with (A). *Then* (D) *can be refined to a decomposition centrally isomorphic with* (A).

This theorem follows immediately from Theorem (5.10) and proposition (5.13).

A special case of (5.14) is the following theorem:

(5.15) Theorem. *If two decompositions of a group into direct sum of indecomposable subgroups are given, and if one of them is consistent with the other, then they are centrally isomorphic.*

§6

(6.1) Theorem. *Let L be a partially ordered set,*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

a direct decomposition of a group G, F a subset of L satisfying the minimal condition, and $\{C_\lambda\}_{\lambda \in F}$ the set of subgroups of G satisfying the condition

$$C_\lambda + \sum_{i \in U_\lambda} E_i = \sum_{i \in V_\lambda} E_i \quad (\lambda \in F), \tag{S}$$

where V_λ is the conbinatorial closure of λ in L and $U_\lambda = V_\lambda - \{\lambda\}$. Then for every closed subset P of L the following direct decomposition holds:

$$\sum_{\lambda \in P} E_\lambda = \sum_{\lambda \in P \cap F} C_\lambda + \sum_{\lambda \in P-F} E_\lambda. \tag{b_P}$$

Proof. 1. Let P be a closed subset of L. By Ψ we denote the collection of sets Γ satisfying the following two conditions:

Γ is a closed subset of L contained in P, $\qquad (a_\Gamma)$

$$\sum_{\lambda \in \Gamma} E_\lambda = \sum_{\lambda \in \Gamma \cap F} C_\lambda + \sum_{\lambda \in \Gamma-F} E_\lambda. \tag{b_Γ}$$

It is not difficult to see that the union of an ascending sequence of sets belonging to Ψ is also an element of Ψ. Therefore, by Kuratowski's fundamental theorem on saturated sets*, we deduce that there exists a maximal element T in Ψ, that is, an element such that $\Gamma \in \Psi$ and $T \subset \Gamma$ implies $\Gamma = T$. Thus, there exists a set T satisfying the conditions:

T is a closed subset of L contained in P $\quad (\mathrm{a}_T)$

$$\sum_{\lambda \in T} E_\lambda = \sum_{\lambda \in T \cap F} C_\lambda + \sum_{\lambda \in T - F} E_\lambda \qquad (\mathrm{b}_T)$$

if $\Gamma \in \Psi$ and $T \subset \Gamma$, then $\Gamma = T$. $\quad (\mathrm{c}_T)$

Let us show that $T = P$. This will prove the theorem.

We begin by proving that

$$P \cap F \subset T. \qquad (1)$$

Suppose that (1) does not hold. Then the set $P \cap F - T$ is not empty. By hypothesis, F satisfies the minimal condition. Therefore, there exists at least one minimal element γ among the elements of $P \cap F - T$. We put

$$\Delta = T \cup U_\gamma. \qquad (2)$$

By hypothesis, P is closed in L. Therefore, by (2.8), $U_\gamma \subset P$. Hence, it follows, when the definitions of γ and U_γ are taken into account, that

$$U_\gamma \cap F \subset T. \qquad (3)$$

From (2) and (3) we conclude that

$$\Delta \cap F = T \cap F. \qquad (4)$$

Let us show that $\Delta \in \Psi$. The set Δ is closed in L, because by (2) it is the union of two closed sets. Moreover, $\Delta \subset P$ because $U_\lambda \subset P$ and, by (a_T), $T \subset P$. We shall show that Δ satisfies the condition (b_Δ). By using (b_T) we obtain

$$\sum_{\lambda \in \Delta} E_\lambda = \sum_{\lambda \in T} E_\lambda + \sum_{\lambda \in \Delta - T} E_\lambda =$$

$$\sum_{\lambda \in T \cap F} C_\lambda + \sum_{\lambda \in T - F} E_\lambda + \sum_{\lambda \in \Delta - T} E_\lambda = \sum_{\lambda \in T \cap F} C_\lambda + \sum_{\lambda \in \Delta - F} E_\lambda,$$

so that, by (4), it follows that

$$\sum_{\lambda \in \Delta} E_\lambda = \sum_{\lambda \in \Delta \cap F} C_\lambda + \sum_{\lambda \in \Delta - F} E_\lambda,$$

that is, the set Δ satisfies the condition (b_Δ). This proves that $\Delta \in \Psi$. Moreover, by (2), $T \subset \Delta$. Hence, it follows, by (c_T), that

$$\Delta = T.$$

This equation and (2) show that

$$U_\gamma \subset T. \qquad (5)$$

We put

$$Q = T \cup \{\gamma\} \qquad (6)$$

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*Translator's note: Zorn's Lemma.

and show that $Q \in \Psi$. As $V_\gamma = U_\gamma \cup \{\gamma\}$, we obtain from (5) and (6) that

$$Q = T \cup V_\gamma .$$

Therefore, Q, as the union of two closed sets, is a closed subset of L. Moreover, $Q \subset P$ because $T \subset P$ and $\gamma \in P$. Thus, Q satisfies (a_Q). Let us show that Q satisfies (b_Q). Since $\gamma \in P \cap F - T$, we deduce from (6) that

$$Q \cap F = (T \cap F) \cup \{\gamma\} \tag{7}$$

$$Q - F = T - F. \tag{8}$$

Using (S) and (b_T) and noting that $\gamma \notin T$, we have

$$\sum_{\lambda \in Q} E_\lambda = E_\gamma + \sum_{\lambda \in T} E_\lambda =$$

$$E_\gamma + \sum_{\lambda \in U_\gamma} E_\lambda + \sum_{\lambda \in T-U_\gamma} E_\lambda = C_\gamma + \sum_{\lambda \in U_\gamma} E_\lambda + \sum_{\lambda \in T-U_\gamma} E_\lambda = C_\gamma + \sum_{\lambda \in T} E_\lambda. \tag{9}$$

Furthermore, from (b_T), (7), and (8) we deduce that

$$C_\gamma + \sum_{\lambda \in T} E_\lambda = C_\gamma + \sum_{\lambda \in T \cap F} C_\lambda + \sum_{\lambda \in T-F} E_\lambda =$$

$$\sum_{\lambda \in (T \cap F) \cup \{\gamma\}} C_\lambda + \sum_{\lambda \in T-F} E_\lambda = \sum_{\lambda \in Q \cap F} C_\lambda + \sum_{\lambda \in Q-F} E_\lambda;$$

hence, we obtain, taking (9) into account,

$$\sum_{\lambda \in Q} E_\lambda = \sum_{\lambda \in Q \cap F} C_\lambda + \sum_{\lambda \in Q-F} E_\lambda;$$

that is, Q satisfies (b_Q). Thus, $Q \in \Psi$ and $T \subset Q$. Hence, by (c_T), $T = Q$. But this is impossible because $\gamma \in Q$, and $\gamma \notin T$. This proves that (1) holds.

By (1) we have $P \cap F \subset T \cap F$ and as $T \subset P$,

$$P \cap F = T \cap F. \tag{10}$$

Using (b_T) we obtain

$$\sum_{\lambda \in P} E_\lambda = \sum_{\lambda \in T} E_\lambda + \sum_{\lambda \in P-T} E_\lambda =$$

$$\sum_{\lambda \in T \cap F} C_\lambda + \sum_{\lambda \in T-F} E_\lambda + \sum_{\lambda \in P-T} E_\lambda = \sum_{\lambda \in T \cap F} C_\lambda + \sum_{\lambda \in P-F} E_\lambda;$$

hence, by (10)

$$\sum_{\lambda \in P} E_\lambda = \sum_{\lambda \in P \cap F} C_\lambda + \sum_{\lambda \in P-F} E_\lambda;$$

that is, P satisfies (b_P). Thus, $P \subset \Psi$ and $T \subset P$ and so, by (c_T), $T = P$, so that (b_T) turns into the required direct decomposition (b_P). This completes the proof.

(6.2) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, *consistent with the decomposition*

$$G = \sum_{\nu \in N} A_\nu . \tag{A}$$

Further, let F *be a closed subset of the partially ordered set* $L(\mathrm{E}, \Omega_\Lambda)$ *satisfying the minimal condition, and* $\{C_\lambda\}_{\lambda \in F}$ *the set of subgroups of* G *satisfying the conditions:*

$$C_\lambda + \check{E}_\lambda = E_\lambda + \check{E}_\lambda \qquad (\lambda \in F,\ \check{E}_\lambda = V(E_\lambda, \mathrm{E}, \Omega_\Lambda)), \tag{c}$$

$$C_\lambda \subset A_\nu \qquad (\lambda \in F), \tag{d}$$

where ν *is a fixed element of* N.

Then for any closed subset P *of* $L(\mathrm{E}, \Omega_\Lambda)$ *the following direct decomposition holds*

$$A_\nu \cap H_P = \sum_{\lambda \in P \cap F} C_\lambda + A_\nu \cap H_{P-F}, \tag{e}$$

where $H_P = \sum_{\lambda \in P} E_\lambda$ *and* $H_{P-F} = \sum_{\lambda \in P-F} E_\lambda$.

Proof. Let P be a closed subset of $L(\mathrm{E}, \Omega_\Lambda)$. By V_λ we denote the combinatorial closure of λ in $L(\mathrm{E}, \Omega_\Lambda)$ and we put $U_\lambda = V_\lambda - \{\lambda\}$. Then by (2.9)

$$\bar{E}_\lambda = \sum_{\alpha \in V_\lambda} E_\alpha, \quad \check{E}_\lambda = \sum_{\alpha \in U_\lambda} E_\alpha.$$

On the basis of these equations (c) can be written in the form

$$C_\lambda + \sum_{\alpha \in U_\lambda} E_\alpha = \sum_{\alpha \in V_\lambda} E_\alpha \qquad (\lambda \in F). \tag{f}$$

In (E) we put $L = L(\mathrm{E}, \Omega_\Lambda)$. By hypothesis the set $\{C_\lambda\}_{\lambda \in F}$ satisfies (c) and, therefore, also (f). Moreover, F satisfies the minimal condition. Therefore, by Theorem (6.1) we conclude that

$$H_P = \sum_{\lambda \in P \cap F} C_\lambda + H_{P-F}. \tag{1}$$

Further, by (d) $\sum_{\lambda \in P \cap F} C_\lambda$ is a subgroup of A_ν. Therefore, the direct decomposition (e) follows from (1) by (3.10).

(6.3) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decompositon of a group G, *fully consistent with the decomposition*

$$G = \sum_{\nu \in N} A_\nu, \tag{A}$$

and let F *be a subset of* $L(\mathrm{E}, \Omega_\Lambda)$ *such that for every* $\nu \in N$ *the set* $F_\nu = F \cap S(A_\nu, \mathrm{E}, \Omega_\Lambda)$ *satisfies the minimal condition. Further, let* $\{C_\lambda\}_{\lambda \in F}$ *be the set of subgroups of* G *satisfying the following conditions*[*]*:*

$$C_\lambda + \check{E}_\lambda = E_\lambda + \check{E}_\lambda \qquad (\lambda \in F,\ \check{E}_\lambda = U(E_\lambda, \mathrm{E}, \Omega_\Lambda)), \tag{a}$$

$$C_\lambda \subset A_{\rho(\lambda)} \qquad (\lambda \in F). \tag{b}$$

Then for every closed subset P *of* $L(\mathrm{E}, \Omega_\Lambda)$ *the following direct decomposition*

[*] The symbol ρ denotes the mapping of L onto N defined in proposition (5.8).

holds:

$$H_P = \sum_{\lambda \in P \cap F} C_\lambda + \sum_{\nu \in N} (A_\nu \cap H_{P-F_\nu}); \tag{c}$$

in particular

$$G = \sum_{\lambda \in F} C_\lambda + \sum_{\nu \in N} (A_\nu \cap H_{L-F_\nu}) \tag{C}$$

and (E) *is fully consistent with* (C).

Proof. 1. Let P be a closed subset of $L(\mathrm{E}, \Omega_\Lambda)$. By hypothesis (E) is consistent with (A), F_ν satisfies the minimal conditions (a) and (b). Therefore, by (6.2) we have

$$A_\nu \cap H_P = \sum_{\lambda \in P \cap F_\lambda} C_\lambda + A_\nu \cap H_{P-F_\nu} \quad (\nu \in N). \tag{1}$$

As (E) is consistent with (A), we have by (3.6)

$$H_P = \sum_{\nu \in N} (A_\nu \cap H_P). \tag{2}$$

It is easy to see that

$$\bigcup_{\nu \in N} (P \cap F_\nu) = P \cap F. \tag{3}$$

For by (5.5) $L = \bigcup_{\nu \in N} S(A_\nu, \mathrm{E}, \Omega_\Lambda)$. Therefore,

$$\bigcup_{\nu \in N} (P \cap F_\nu) = \bigcup_{\nu \in N} (P \cap F \cap S(A_\nu, \mathrm{E}, \Omega_\Lambda)) = P \cap F \cap L = P \cap F.$$

Now when we replace the direct summands in (2) by their expressions (1) and take (3) into account, we obtain

$$H_P = \sum_{\lambda \in P \cap F} C_\lambda + \sum_{\nu \in N} (A_\nu \cap H_{P-F_\nu}). \tag{c}$$

Putting $P = L$ in (c) we obtain the direct decomposition

$$G = \sum_{\lambda \in F} C_\lambda + \sum_{\nu \in N} (A_\nu \cap H_{L-F_\nu}). \tag{C}$$

2. We shall prove now that (E) is consistent with (C).

From (c) and (C) we deduce, with the help of the relation $H_{P-F_\nu} \subset H_{L-F_\nu}$, that

$$C_i \cap H_P \subset C_i \cap \Big(\sum_{\alpha \in F-\{i\}} C_\alpha + \sum_{\nu \in N} (A_\nu \cap H_{L-F_\nu})\Big) = \{0\} \quad (i \in F-P);$$

hence,

$$C_i \cap H_P = \{0\} \quad (i \in F-P). \tag{4}$$

Moreover, by (c)

$$C_i \cap H_P = C_i \quad (i \in F \cap P). \tag{5}$$

Furthermore, as $(L - F_\nu) \cap P = P - F_\nu$, we have

$$(A_\nu \cap H_{L-F_\nu}) \cap H_P = A_\nu \cap H_{P-F_\nu} \quad (\nu \in N). \tag{6}$$

Now by (4), (5), (6) the equation (c) can be written in the form

$$H_P = \sum_{\lambda \in F} (C_\lambda \cap H_P) + \sum_{\nu \in N} ((A_\nu \cap H_{L-F_\nu}) \cap H_P). \tag{7}$$

Since (7) holds for every closed subset P of $L(\mathrm{E}, \Omega_{\mathrm{A}})$, the decomposition (E) is consistent with (C) by (3.7).

3. We shall show now that a subset P of L is closed in $L(\mathrm{E}, \Omega_{\mathrm{C}})$ if and only if it is closed in $L(\mathrm{E}, \Omega_{\mathrm{A}})$.

We have proved above that (7) holds for every closed subset P of $L(\mathrm{E}, \Omega_{\mathrm{A}})$; by (3.6), therefore, P is also closed in $L(\mathrm{E}, \Omega_{\mathrm{C}})$.

Let us assume, conversely, that P is a closed subset of $L(\mathrm{E}, \Omega_{\mathrm{C}})$. We shall show that the following direct decomposition holds:

$$H_P = \sum_{\nu \in N} (A_\nu \cap H_P); \tag{8}$$

this will prove, by (3.6), that P is a closed subset of $L(\mathrm{E}, \Omega_{\mathrm{A}})$. As (E) is consistent with (C) and P is a closed subset of $L(\mathrm{E}, \Omega_{\mathrm{C}})$, we have, by (3.6), the following direct decomposition

$$H_P = \sum_{\lambda \in F} (C_\lambda \cap H_P) + \sum_{\nu \in N} ((A_\nu \cap H_{L-F_\nu}) \cap H_P).$$

Since by hypothesis $C_\lambda \subset A_{\rho(\lambda)}$ for $\lambda \in F$, it is not difficult to see that we obtain from this direct decomposition the relation

$$H_P \subset \sum_{\nu \in N} (A_\nu \cap H_P)$$

from which (8) follows, because the converse inclusion evidently holds.

4. By V_λ we denote the combinatorial closure of λ in $L(\mathrm{E}, \Omega_{\mathrm{A}})$, and we put $U_\lambda = V_\lambda - \{\lambda\}$. Then, by (2.9)

$$\overline{E}_\lambda = H_{V_\lambda} = \sum_{i \in V_\lambda} E_i\,, \tag{9}$$

$$\overline{E}_\lambda = H_{U_\lambda} = \sum_{i \in U_\lambda} E_i\,. \tag{10}$$

On the strength of what we have proved under 3 we can assert that V_λ is the combinatorial closure of λ also in $L(\mathrm{E}, \Omega_{\mathrm{C}})$. Therefore, $\overline{E}_\lambda = U(E_\lambda, \mathrm{E}, \Omega_{\mathrm{A}})$ is also the least group of $\Delta(\mathrm{E}, \Omega_{\mathrm{C}})$ containing E_λ, that is,

$$\overline{E}_\lambda = U(E_\lambda, \mathrm{E}, \Omega_{\mathrm{C}}) \qquad (\lambda \in L); \tag{11}$$

and $\check{E}_\lambda = V(E_\lambda, \mathrm{E}, \Omega_{\mathrm{A}})$ is the largest subgroup of $\overline{E}_\lambda$ belonging to $\Delta(\mathrm{E}, \Omega_{\mathrm{C}})$ and not containing E_λ, that is,

$$\check{E}_\lambda = V(E_\lambda, \mathrm{E}, \Omega_{\mathrm{C}}) \qquad (\lambda \in L). \tag{12}$$

5. We shall now show that the following relations hold:

$$(H_{L-F_\nu} \cap A_\nu) \cap \overline{E}_\lambda \subset \check{E}_\lambda \qquad (\nu \neq \rho(\lambda),\ \nu \in N,\ \lambda \in L), \tag{13}$$

$$(H_{L-F_\nu} \cap A_\nu) \cap E_\lambda \subset \check{E}_\lambda \qquad (\nu \in N,\ \lambda \in F_\nu), \tag{14}$$

$$C_i \cap \overline{E}_\lambda \subset \check{E}_\lambda \qquad (i \neq \lambda,\ i \in F,\ \lambda \in L). \tag{15}$$

By hypothesis (E) is fully consistent with (A). Therefore, by (5.9) we have

$$A_\lambda \cap \overline{E}_\lambda = A_\nu \cap \check{E}_\lambda \qquad (\nu \neq \rho(\lambda),\ \nu \in N,\ \lambda \in L),$$

from which (13) follows.

Taking (9) and (10) into account it is easy to see that

$$H_{L-F_\nu} \cap \overline{E}_\lambda = H_{L-F_\nu} \cap H_{V_\lambda} = H_{V_\lambda - F_\nu},\ H_{L-F_\nu} \cap \check{E}_\lambda = H_{U_\lambda - F_\nu}.$$

Moreover, evidently $V_\lambda - F_\nu = U_\lambda - F_\nu$ for $\lambda \in F_\nu$. Therefore,

$$H_{L-F_\nu} \cap \overline{E}_\lambda = H_{L-F_\nu} \cap \check{E}_\lambda \qquad (\lambda \in F_\nu,\ \nu \in N),$$

from which (14) follows.

Finally, putting $P = U_\lambda$ in (4) and (5) and replacing H_{U_λ} by $\check{E}_\lambda$, we obtain the relations:

$$C_i \cap \check{E}_\lambda = \{0\} \qquad (i \in F - U_\lambda),$$

$$C_i \cap \check{E}_\lambda = C_i \qquad (i \in F \cap U_\lambda),$$

from which (15) follows.

6. We shall show now that (E) is fully consistent with (C). That (E) is consistent with (C) has been proved above. According to definition (5.7) it is, therefore, sufficient to show that (C) has the property (b) of (5.6). As the relations (12) and (13) hold, it is, therefore, sufficient to prove that for every fixed $\lambda \in L$ the intersection with $\overline{E}_\lambda$ of every direct summand of (C), possibly with one exception, is contained in $\check{E}_\lambda$. Two cases can occur:

First case. $\lambda \in F$. In this case there exists an index $\nu \in N$ such that $\lambda \in F_\nu$, because $F = \bigcup_{\nu \in N} F_\nu$. From (13), (14), and (15) we deduce, therefore, that the intersection with $\overline{E}_\lambda$ of every direct summand of (C), except–possibly– C_λ, is contained in $\check{E}_\lambda$.

Second case. $\lambda \notin F$. In this case we deduce from (13) and (15) that the intersection with $\overline{E}_\lambda$ of every direct summand of (C), except–possibly–

$$A_{\rho(\lambda)} \cap H_{L-F_{\rho(\lambda)}},$$

is contained in $\check{E}_\lambda$.

Thus, (E) is fully consistent with (C). This completes the proof.

(6.4) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, fully consistent with the decomposition

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

and let F be a subset of $L(\mathrm{E}, \Omega_A)$ such that for every $\nu \in N$ the set F_ν, $F_\nu = F \cap S(A_\nu, \mathrm{E}, \Omega_A)$ satisfies the minimal condition. By C_λ, $\lambda \in L$, we denote the *component of* E_λ *in the direct summand* $A_{\rho(\lambda)}$ *in* (A). *Then for every closed subset* P *of* $L(\mathrm{E}, \Omega_{\mathrm{A}})$ *we have the direct decomposition*

$$H_P = \sum_{\lambda \in P \cap F} C_\lambda + \sum_{\nu \in N} (A_\nu \cap H_{P-F_\nu})$$

and, in particular,

$$G = \sum_{\lambda \in F} C_\lambda + \sum_{\nu \in N} (A_\nu \cap H_{L-F_\nu}).$$

Proof. By (5.11j) C_λ satisfies the condition

$$C_\lambda + \check{E}_\lambda = E_\lambda + \check{E}_\lambda \qquad (\lambda \in F). \tag{c}$$

Moreover, as C_λ is the component of E_λ in the direct summand $A_{\rho(\lambda)}$ of (A), C_λ satisfies the condition

$$C_\lambda \subset A_{\rho(\lambda)} \qquad (\lambda \in F). \tag{d}$$

Therefore, proposition (6.4) is a consequence of proposition (6.3).

(6.5) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, *consistent with the decomposition*

$$G = \sum_{\nu \in N} A_\nu. \tag{A}$$

Then the following relations hold:

$$A_\nu \cap H_{L-S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})} = \{0\}, \qquad (\nu \in N). \tag{1}$$

Proof. Suppose that there is an index ν in N for which (1) does not hold. Then there exists an element x in G, $x \neq 0$, such that

$$x \in A_\nu, \tag{2}$$

$$x \in H_{L-S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})}. \tag{3}$$

By Γ we denote the set of those indices $\lambda \in L$ for which x has a non-null component in the direct summand E_λ of (E). By (3)

$$\Gamma \subset L - S(A_\nu, \mathrm{E}, \Omega_{\mathrm{A}}). \tag{4}$$

Since x is not the null element, Γ is a finite *non-empty* set. By γ we denote an arbitrary maximal element of Γ which we regard here as a subset of $L(\mathrm{E}, \Omega_{\mathrm{A}})$. Such an element exists, because Γ is a finite non-empty set. By (4)

$$\gamma \notin S(A_\nu, \mathrm{E}, \Omega_{\mathrm{A}}). \tag{5}$$

By V_γ we denote the combinatorial closure of γ in $L(\mathrm{E}, \Omega_{\mathrm{A}})$ and we put $U_\gamma = V_\gamma - \{\gamma\}$. Further, we denote by P the set of all elements of L that are greater than γ or incomparable with γ, and we put

$$Q = P \cup \{\gamma\}.$$

It is easy to see that the following relations hold:

$$\Gamma \subset Q, \tag{6}$$

$$\Gamma \not\subset P, \tag{7}$$

$$U_\gamma \subset P, \tag{8}$$

$$Q = P \cup V_\gamma . \tag{9}$$

From the definition of Γ it follows by (6) and (7) that

$$x \in H_Q,\ x \notin H_P;$$

hence, we obtain from (2)

$$A_\nu \cap H_Q \neq A_\nu \cap H_P. \tag{10}$$

From (9) and (10) follows

$$H_{U_\gamma} \subset H_P, \tag{11}$$

$$H_Q = [H_P, H_{V_\gamma}]. \tag{12}$$

The sets P and V_γ are closed in $L(\mathrm{E}, \Omega_\mathrm{A})$, as follows immediately from their definitions. By (9) Q is also closed in $L(\mathrm{E}, \Omega_\mathrm{A})$. Moreover, by hypothesis (E) is consistent with (A). By (2.5) the subgroups H_P, H_{V_γ}, H_Q are, therefore, admissible with respect to the set of endomorphisms Ω_A. Hence, by (3.9) we have

$$A_\nu \cap H_Q = [A_\nu \cap H_P, A_\nu \cap H_{V_\gamma}]. \tag{13}$$

Now it is easy to show that

$$A_\nu \cap H_{V_\gamma} \neq A_\nu \cap H_{U_\gamma}. \tag{14}$$

For if we assume that

$$A_\nu \cap H_{V_\gamma} = A_\nu \cap H_{U_\gamma},$$

then we obtain from (11)

$$A_\nu \cap H_{V_\gamma} \subset A_\nu \cap H_P,$$

hence by (13)

$$A_\nu \cap H_Q = A_\nu \cap H_P,$$

which is impossible by (10). Thus, (14) holds.

Taking into account that by (2.9) $H_{V_\gamma} = \overline{E}_\gamma$ and $H_{U_\gamma} = \check{E}_\gamma$ we can write (14) in the form

$$A_\nu \cap \overline{E}_\gamma \neq A_\nu \cap \check{E}_\gamma;$$

hence, we obtain

$$A_\nu \cap \overline{E}_\gamma \not\subset \check{E}_\gamma . \tag{15}$$

From (15) it follows according to the definition (5.4) that

$$\gamma \in S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}}). \tag{16}$$

But (16) contradicts (5). This proves that (1) holds for every $\nu \in N$.

Proposition (6.5) is now proved.

(6.6) **Theorem.** *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, fully consistent with the decomposition

$$G = \sum_{\nu \in N} A_\nu. \tag{A}$$

By C_λ, $\lambda \in L$, we denote the component of E_λ in the direct summand $A_{\rho(\lambda)}$ of (A). *If the subset $S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})$ of $L(\mathrm{E},\ \Omega_{\mathrm{A}})$ satisfies the minimal condition, then the following direct decomposition holds:*

$$A_\nu = \sum_{\lambda \in S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})} C_\lambda. \tag{C$_\nu$}$$

Proof. As C_λ is the component of E_λ in the direct summand $A_{\rho(\lambda)}$ of (A), it satisfies the condition

$$C_\lambda \subset A_{\rho(\lambda)} \qquad (\lambda \in S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})),$$

and, by (5.11), also the condition

$$C_\lambda + \check{E}_\lambda = E_\lambda + \check{E}_\lambda \qquad (\lambda \in L,\ \check{E}_\lambda = V(E_\lambda,\ \mathrm{E},\ \Omega_{\mathrm{A}})). \tag{c}$$

But, by hypothesis, $S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})$ satisfies the minimal condition. Therefore, putting $F = S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})$ we can assert, on the strength of (6.2) that

$$A_\nu \cap H_P = \sum_{\lambda \in P \cap S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})} C_\lambda + A_\nu \cap H_{P - S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})}. \tag{e}$$

When we put $P = L$ in (e) and note that $A_\nu \cap H_L = A_\nu \cap G = A_\nu$, we obtain

$$A_\nu = \sum_{\lambda \in S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})} C_\lambda + A_\nu \cap H_{L - S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})}. \tag{1}$$

But by (6.5)

$$A_\nu \cap H_{L - S(A_\nu,\ \mathrm{E},\ \Omega_{\mathrm{A}})} = \{0\}, \tag{2}$$

because (E) is, by hypothesis, consistent with (A). Now comparing (1) and (2) we see that (C$_\nu$) holds.

§7

Let $G = \sum_{\lambda \in L} E_\lambda$ and let Γ be an arbitrary subset of L. Throughout the following we shall denote the subgroup $\sum_{\lambda \in \Gamma} E_\lambda$ of G briefly by H_Γ.

(7.1) **Theorem.** *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, fully consistent with the decomposition

$$G = \sum_{\nu \in N} A_\nu, \tag{A}$$

and let F be a finite subset of L. Then for every closed subset P of $L(\mathrm{E}, \Omega_\mathrm{A})$ the following direct decomposition holds:

$$H_P = H_{P \cap F} + \sum_{\nu \in N} (A_\nu \cap H_{P-F_\nu}), \tag{a_P}$$

where $F_\nu = F \cap S(A_\nu, \mathrm{E}, \Omega_\mathrm{A})$.

Proof. Let F be a finite subset of L and P a closed subset of $L(\mathrm{E}, \Omega_\mathrm{A})$. We shall prove that ($a_P$) holds. The proof will be by induction over the cardinal number of F. The theorem is, obviously, true when F is the empty set. Let us assume that it is true for every set F' having a smaller cardinal number than F and show that it is then also true for F.

Two cases are possible: $F \not\subset P$, $F \subset P$.

In the first case there exists an element $\gamma \in F - P$. We put $F' = F - \{\gamma\}$. Since F' has a smaller cardinal number than F, we have by induction hypothesis

$$H_P = H_{P \cap F'} + \sum_{\nu \in N} (A_\nu \cap H_{P-F'_\nu}), \text{ where } F'_\nu = F' \cap S(A_\nu, \mathrm{E}, \Omega_\mathrm{A}). \tag{*}$$

It is easy to see that the relations $\gamma \in F - P$, $F' = F - \{\gamma\}$ imply that $P \cap F' = P \cap F$, $P - F'_\nu = P - F_\nu$ so that (a_P) follows from (*). Thus, (a_P) holds when $F \not\subset P$. Throughout the following we shall, therefore, assume that the second case holds, that is,

$$F \subset P. \tag{1}$$

We shall regard F as a direct subset of $L(\mathrm{E}, \Omega_\mathrm{A})$. By a we denote an arbitrary maximal element F. As F is not the empty set, such an element exists. We put $F^* = F - \{a\}$. Then

$$F = F^* \cup \{a\} \qquad a \notin F^*. \tag{2}$$

By Q we denote the least closed subset of $L(\mathrm{E}, \Omega_\mathrm{A})$ containing F^*. Since P is a closed set containing F^*, we have

$$Q \subset P. \tag{3}$$

When we take into account that a is a maximal element of F and that $a \notin F^*$, we easily see that

$$a \notin Q. \tag{4}$$

By V_a we denote the combinatorial closure of a in $L(\mathrm{E}, \Omega_\mathrm{A})$ and we put

$$U_a = V_a - \{a\}. \tag{5}$$

It is easy to see that U_a is a closed set contained in P. We put

$$\Gamma = Q \cup U_a, \tag{6}$$

then

$$\Gamma \subset P, \tag{7}$$

since $U_a \in P$ and, by (3), $Q \subset P$. The set Γ, as the union of two closed sets, is closed. As $F^* \subset Q$ it follows from (6) that

$$F^* \subset \Gamma. \tag{8}$$

Further, from (4), (5), and (6) we deduce that

$$a \notin \Gamma, \tag{9}$$

hence, (2) and (8) imply that

$$F^* = \Gamma \cap F. \tag{10}$$

We put

$$F_\nu^* = F^* \cap S(A_\nu, \mathrm{E}, \Omega_{\mathrm{A}}) \tag{11}$$

$$F_\nu = F \cap S(A_\nu, \mathrm{E}, \Omega_{\mathrm{A}}). \tag{12}$$

From (10), (11), and (12) we deduce that

$$\Gamma - F_\nu^* = \Gamma - F_\nu \qquad (\nu \in N). \tag{13}$$

By (2) the set F^* has a smaller cardinal number than F so that, by induction hypothesis, we have

$$H_\Gamma = H_{\Gamma \cap F^*} + \sum_{\nu \in N} (A_\nu \cap H_{\Gamma - F_\nu^*}).$$

Hence (8) and (13) imply that

$$H_\Gamma = H_{F^*} + \sum_{\nu \in N} (A_\nu \cap H_{\Gamma - F_\nu^*}). \tag{14}$$

By C_λ we denote the component of E_λ in the direct summand $A_{\rho(\lambda)}$ of (A). By hypothesis (E) is fully consistent with (A) and F is a finite subset of L. Moreover, Γ and P are closed subsets of $L(\mathrm{E}, \Omega_{\mathrm{A}})$. Therefore, we have, by (6.4)

$$H_\Gamma = \sum_{\lambda \in \Gamma \cap F} C_\lambda + \sum_{\nu \in N} (A_\nu \cap H_{\Gamma - F_\nu}), \tag{15}$$

$$H_P = \sum_{\lambda \in P \cap F} C_\lambda + \sum_{\nu \in N} (A_\nu \cap H_{P - F_\nu}). \tag{16}$$

The group $A_\nu \cap H_{\Gamma - F_\nu}$ is a direct summand of G, because it is, by (15) a direct summand of H_Γ, and H_Γ is, evidently, a direct summand of G. Moreover, $A_\nu \cap H_{\Gamma - F_\nu} \subset A_\nu \cap H_{P - F_\nu}$, because, by (7), $\Gamma \subset P$. Therefore, $A_\nu \cap H_{\Gamma - F_\nu}$ is a direct summand of $A_\nu \cap H_{P - F_\nu}$

$$A_\nu \cap H_{P - F_\nu} = A_\nu \cap H_{\Gamma - F_\nu} + X_\nu \qquad (\nu \in N). \tag{17}$$

From (1), (2), and (8) we conclude that $P \cap F = \{a\} \cup \Gamma \cap F$. Therefore, taking (15) and (17) into account we can write (16) in the form

$$H_P = C_\alpha + H_\Gamma + \sum_{\nu \in N} X_\nu. \tag{18}$$

We shall now prove that

$$C_\alpha + H_\Gamma = E_\alpha + H_\Gamma. \tag{19}$$

As (E) is, by hypothesis, fully consistent with (A), we have by (5.11j) $C_\alpha + \check{E}_\alpha = E_\alpha + \check{E}_\alpha$, where $\check{E}_\alpha = \sum_{\lambda \in U_\alpha} E_\lambda$, that is,

$$C_\alpha + \sum_{\lambda\in U_\alpha} E_\lambda = E_\alpha + \sum_{\lambda\in U_\alpha} E_\lambda. \tag{20}$$

By (6) and (9) $U_\alpha \subset \Gamma$ and $\alpha\in\Gamma$. Hence, (20) implies that

$$C_\alpha + \sum_{\lambda\in\Gamma} E_\lambda = E_\alpha + \sum_{\lambda\in\Gamma} E_\lambda,$$

and (19) follows, because $\sum_{\lambda\in\Gamma} E_\lambda = H_\Gamma$.

Comparing (18) and (19) we obtain

$$H_P = E_\alpha + H_\Gamma + \sum_{\nu\in N} X_\nu;$$

hence, by (14)

$$H_P = E_\alpha + H_F + \sum_{\nu\in N} (A_\nu \cap H_{\Gamma - F_\nu}) + \sum_{\nu\in N} X_\nu.$$

This equation can, by (17), be written in the form

$$H_P = E_\alpha + H_{F^*} + \sum_{\nu\in N} (A_\nu \cap H_{P-F_\nu}). \tag{21}$$

By (2), $E_\alpha + H_{F^*} = H_F$. Also, by (1), $F = P \cap F$. Therefore,

$$E_\alpha + H_{F^*} = H_{P\cap F}. \tag{22}$$

Comparing (21) and (22) we obtain

$$H_P = H_{P\cap F} + \sum_{\nu\in N} (A_\nu \cap H_{P-F_\nu}).$$

Thus, (a_P) also holds in the second case, when $F\subset P$. This completes the proof of Theorem (7.1).

(7.2) *Let*

$$G = \sum_{\lambda\in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, fully consistent with the decomposition

$$G = \sum_{\lambda\in N} A_\nu. \tag{A}$$

Further, let A_k be a direct summand of (A) *and x an element of A_k, $x\neq 0$. Then there exists a finite subset Γ of $S(A_k, \mathrm{E}, \Omega_A)$ and a set $\{C_\lambda\}_{\lambda\in\Gamma}$ of subgroups of A_k satisfying the following conditions;*

$$C_\lambda + \check{E}_\lambda = E_\lambda + \check{E}_\lambda \quad (\lambda\in\Gamma,\ \check{E}_\lambda = U(E_\lambda, \mathrm{E}, \Omega_A)), \tag{a}$$

$$A_k = \sum_{\lambda\in\Gamma} C_\lambda + A_k \cap H_{L-\Gamma}, \tag{b}$$

$$x \in \sum_{\lambda\in\Gamma} C_\lambda; \tag{c}$$

(d)
the following direct decomposition holds:

$$G = \sum_{\lambda\in\Gamma} C_\lambda + A_k \cap H_{L-\Gamma} + \sum_{\nu\in N-\{k\}} A_\nu \tag{C}$$

and (E) *is fully consistent with* (C).

Proof. By hypothesis x is a non-null element of G. Therefore, there exists a non-empty finite subset F of L satisfying the condition

$$x \in H_F, \text{ where } H_F = \sum_{\lambda \in F} E_\lambda. \tag{1}$$

By V_λ we denote the conbinatorial closure of λ in $L(\mathrm{E}, \Omega_\mathrm{A})$ and we put $U_\lambda = V_\lambda - \{\lambda\}$. By hypothesis (E) is fully consistent with (A). Therefore, we deduce from (1) that the following direct decompositions hold:

$$G = H_F + \sum_{\nu \in N} (A_\nu \cap H_{L - F_\nu}) \tag{2}$$

$$H_{V_\lambda} = H_{V_\lambda \cap F} + \sum_{\nu \in N} (A_\nu \cap H_{V_\lambda - F_\nu} \qquad (\lambda \in F_k), \tag{3}$$

$$H_{U_\lambda} = H_{U_\lambda \cap F} + \sum_{\nu \in N} (A_\nu \cap H_{U_\lambda - F_\nu} \qquad (\lambda \in F_k), \tag{4}$$

where $F_\nu = F \cap S(A_\nu, \mathrm{E}, \Omega_\mathrm{A})$. These equations (2), (3), and (4) are obtained from (a_P) of Theorem (7.1) when we put P equal to L, U_λ, and V_λ, respectively. By (3.10), (2), (3), and (4) imply:

$$A_k = A_k \cap H_{L-F_k} + X, \tag{5}$$

$$A_k \cap H_{V_\lambda} = A_k \cap H_{V_\lambda - F_k} + X_\lambda \qquad (\lambda \in F_k), \tag{6}$$

$$A_k \cap H_{U_\lambda} = A_k \cap H_{U_\lambda - F_k} + Y_\lambda \qquad (\lambda \in F_k), \tag{7}$$

where

$$X = A_k \cap (H_F + \sum_{\nu \in N - \{k\}} (A_\nu \cap H_{L - F_\nu})), \tag{8}$$

$$X_\lambda = A_k \cap (H_{V_\lambda \cap F} + \sum_{\nu \in N - \{k\}} (A_\nu \cap H_{V_\lambda - F_\nu})) \qquad (\lambda \in F_k), \tag{9}$$

$$Y_\lambda = A_k \cap (H_{U_\lambda \cap F} + \sum_{\nu \in N - \{k\}} (A_\nu \cap H_{U_\lambda - F_\nu})) \qquad (\lambda \in F_k). \tag{10}$$

By (7), Y_λ is a direct summand of $A_k \cap H_{U_\lambda}$. By hypothesis (E) is consistent with (A) so that, by (4.6), $H_{U_\lambda} = \sum_{\lambda \in N} (A_\nu \cap H_{U_\lambda}$ and $A_k \cap H_{U_\lambda}$ is a direct summand of H_{U_λ} and, hence, of G. Therefore, Y_λ is a direct summand of G. Further (9) and (10) show that $Y_\lambda \subset X_\lambda$. Therefore, Y_λ is a direct summand of X_λ, that is, there exists a subgroup C_λ of X_λ such that

$$X_\lambda = Y_\lambda + C_\lambda \qquad (\lambda \in F_k). \tag{11}$$

From (6), (7) and (11) we deduce that

$$A_k \cap H_{V_\lambda} = C_\lambda + A_k \cap H_{U_\lambda} \qquad (\lambda \in F_k).$$

As $\overline{E}_\lambda = H_{V_\lambda}$ and $\check{E}_\lambda = H_{U_\lambda}$, this relation can be written in the form

$$A_k \cap \overline{E}_\lambda = C_\lambda + A_k \cap \check{E}_\lambda \qquad (\lambda \in F_k). \tag{12}$$

By hypothesis, (E) is fully consistent with (A). Therefore, by (5.11f)

$$[A_k \cap \overline{E}_\lambda, \check{E}_\lambda] = \overline{E}_\lambda \qquad (\lambda \in F_k). \tag{13}$$

From (12) and (13) we conclude that $C_\lambda + \check{E}_\lambda = \overline{E}_\lambda$, and since $\overline{E}_\lambda = E_\lambda + \check{E}_\lambda$, we obtain

$$C_\lambda + \check{E}_\lambda = E_\lambda + \check{E}_\lambda, \qquad (\lambda \in F_k). \tag{14}$$

Further, from (9) and (11) it follows that

$$C_\lambda \subset A_k \qquad (\lambda \in F_k). \tag{15}$$

The relations (14) and (15) show that the set $\{C_\lambda\}_{\lambda \in F_k}$ of subgroups of G satisfies the conditions (c) and (d) of proposition (6.2). Moreover, F_k is a finite subset of L and, by hypothesis, (E) is consistent with (A). Therefore, by (6.2), we have the direct decomposition

$$A_k = \sum_{\lambda \in F_k} C_\lambda + A_k \cap H_{L-F_k}. \tag{16}$$

(16) is obtained from equation (e) of (6.2) by putting $F = F_k$ and $P = L$.

By (11) $C_\lambda \subset X_\lambda$, and (8) and (9) show that $X_\lambda \subset X$. Hence, $C_\lambda \subset X$ for $\lambda \in F_k$. From this follows

$$\sum_{\lambda \in F_k} C_\lambda \subset X. \tag{17}$$

Now we deduce from (5), (16), and (17) that

$$X = \sum_{\lambda \in F_k} C_\lambda. \tag{18}$$

Further, by hypothesis $x \in A_k$ and, by (1), $x \in H_F$. Hence, it follows by (8) that $x \in X$ and so, by (18)

$$x \in \sum_{\lambda \in F_k} C_\lambda. \tag{19}$$

We put $\Gamma = F_k$. Then (14), (16), and (19) show that the set $\{C_\lambda\}_{\lambda \in \Gamma}$ of subgroups C_λ satisfies the conditions (a),(b),(c). Finally, the decomposition (C) holds on account of (A) and (16), and by (6.3). (E) is fully consistent with (C) so that the groups C_λ also satisfy condition (d).

(7.3) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, fully consistent with the decomposition

$$G = \sum_{\nu \in N} A_\nu. \tag{A}$$

Suppose, further, that an arbitrary direct summand A_β of (A) *has a direct decomposition*

$$A_\beta = \sum_{\lambda \in Q} C_\lambda, \tag{1}$$

where Q is a subset of L and where the direct summands C_λ satisfy the condition

$$C_\lambda + \check{E}_\lambda = \overline{E}_\lambda \qquad (\lambda \in Q). \tag{2}$$

Then $Q = S(A_\beta, \mathrm{E}, \Omega_\Lambda)$ *and for every closed subset* P *of* $L(\mathrm{E}, \Omega_\Lambda)$ *the following direct decomposition holds:*

$$A_\beta \cap H_P = \sum_{\lambda \in P \cap Q} C_\lambda. \tag{3}$$

Proof. Let P be a closed subset of $L(\mathrm{E}, \Omega_\Lambda)$. We shall show that (3) holds. Let F be a finite subset of $Q - P$,

$$F \subset Q - P. \tag{4}$$

Taking the condition of the theorem into account we deduce from (6.2) that

$$A_\beta \cap H_L = \sum_{\lambda \in F} C_\lambda + A_\beta \cap H_{L-F}. \tag{5}$$

By (4) $F \subset L - P$, hence, $P \subset L - F$ and, therefore,

$$H_P \subset H_{L-F}. \tag{6}$$

From (5) and (6) we deduce that

$$\sum_{\lambda \in F} C_\lambda \cap (A_\beta \cap H_P) \subset \sum_{\lambda \in F} C_\lambda \cap (A_\beta \cap H_{L-F}) = \{0\};$$

hence,

$$\sum_{\lambda \in F} C_\lambda \cap (A_\beta \cap H_P) = \{0\}. \tag{7}$$

As F is an arbitrary finite subset of Q, (7) implies that

$$\sum_{\lambda \in Q-P} C_\lambda \cap (A_\beta \cap H_P) = \{0\}. \tag{8}$$

Since P is a closed subset of $L(\mathrm{E}, \Omega_\Lambda)$ it is easy to see that

$$\overline{E}_\lambda \subset H_P \qquad (\lambda \in P);$$

hence, by (2),

$$C_\lambda \subset H_P \qquad (\lambda \in P \cap Q). \tag{9}$$

Moreover, by (1),

$$C_\lambda \subset A_\beta \qquad (\lambda \in Q). \tag{10}$$

From (1), (9), and (10) we deduce that

$$\sum_{\lambda \in P \cap Q} C_\lambda \subset A_\beta \cap H_P. \tag{11}$$

Since $Q = (Q - P) \cup (P \cap Q)$, (1) can be written in the form

$$A_\beta = \sum_{\lambda \in Q-P} C_\lambda + \sum_{\lambda \in P \cap Q} C_\lambda. \tag{12}$$

Now from (8), (11), and (12) we conclude that (3) holds.

We shall now show that

$$Q = S(A_\beta, \mathrm{E}, \Omega_\Lambda).$$

Let

$$\alpha \in S(A_\beta, \mathrm{E}, \Omega_\Lambda). \tag{13}$$

By V_α we denote the combinatorial closure of α in $L(\mathrm{E}, \Omega_\Lambda)$ and by U_α the set

$V_\alpha - \{\alpha\}$. When we put the set P in (3) equal to V_α and U_α and note that $H_{U_\alpha} = \check{E}_\alpha$, $H_{V_\alpha} = \overline{E}_\alpha$, we obtain

$$A_\beta \cap \check{E}_\alpha = \sum_{\lambda \in U_\alpha \cap Q} C_\lambda, \tag{14}$$

$$A_\beta \cap \overline{E}_\alpha = \sum_{\lambda \in V_\alpha \cap Q} C_\lambda. \tag{15}$$

If $\alpha \in Q$, then $U_\alpha \cap Q = V_\alpha \cap Q$ and therefore, by (14) and (15)

$$A_\beta \cap \overline{E}_\alpha = A_\beta \cap \check{E}_\alpha,$$

which is impossible by (13). This proves that $\alpha \in Q$ if (13) holds. Therefore,

$$Q \supset S(A_\beta, \mathrm{E}, \Omega_\Lambda). \tag{16}$$

Let us assume now that $\alpha \notin S(A_\beta, \mathrm{E}, \Omega_\Lambda)$. Then, by (5.9) $A_\beta \cap \overline{E}_\alpha = A_\beta \cap \check{E}_\alpha$. As the relations (14) and (15) hold for every $\alpha \in L$, we obtain

$$\sum_{\lambda \in U_\alpha \cap Q} C_\lambda = \sum_{\lambda \in V_\alpha \cap Q} C_\lambda.$$

This shows that $U_\alpha \cap Q = V_\alpha \cap Q$ and, therefore, $\alpha \notin Q$. Thus, $Q \subset S(A_\beta, \mathrm{E}, \Omega_\Lambda)$. Comparing this relation with (16) we obtain the equation $Q = S(A_\beta, \mathrm{E}, \Omega_\Lambda)$.

(7.4) Theorem. *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, fully consistent with the decomposition

$$G = \sum_{\nu \in N} A_\nu. \tag{A}$$

If every direct summand A_β of (A) *is a countable or finite group, then there exists a set $\{C_\lambda\}_{\lambda \in L(A_\beta)}$ of subgroups of G with the following properties:*

$$A_\beta = \sum_{\lambda \in S(A_\beta, \mathrm{E}, \Omega_\Lambda)} C_\lambda \tag{1}$$

$$C_\lambda + \check{E}_\lambda = E_\lambda + \check{E}_\lambda \quad (\lambda \in L(A_\beta)). \tag{2}$$

Proof. By hypothesis A_β is a countable or finite group. Therefore, there exists a sequence

$$x_1, x_2, \cdots, x_n, \cdots, \tag{3}$$

containing all non-null elements of A_β, indexed by the natural numbers.

We shall show that for every natural number n there exists a collection $\{T_i\}_{0 < i \leq n}$ of subsets of L and sets of subgroups of G,

$$\{C_\lambda\}_{\lambda \in T_n}, \{A_\beta^{(i)}\}_{0 < i \leq n}$$

with the following properties:

$$C_\lambda + \check{E}_\lambda = \overline{E}_\lambda \quad (\lambda \in T_n), \tag{a$_n$}$$

$$A_\beta = \sum_{\lambda \in T_n} C_\lambda + A_\beta^{(n)}, \tag{$\mathrm{b_n}$}$$

$$\{x_1, x_2, \cdots, x_n\} \subset \sum_{\lambda \in T_n} C_\lambda, \tag{$\mathrm{c_n}$}$$

the following *direct* decomposition holds: $(\mathrm{d_n})$

$$G = \sum_{\lambda \in T_n} C_\lambda + A_\beta^{(n)} + \sum_{\nu \in N-\{\beta\}} A_\nu \tag{$\mathrm{C_n}$}$$

and (E) is fully consistent with $(\mathrm{C_n})$,

$$T_1 \subset T_2 \subset \cdots \subset T_n \subset L. \tag{$\mathrm{e_n}$}$$

By (7.2) the required set exists for $n = 1$. Let us assume that for a certain natural number n the sets

$$\{T_i\}_{0<i\leq n}, \{C_\lambda\}_{\lambda\in T_n}, \{A_\beta^{(i)}\}_{0<i\leq n}$$

with the properties $(\mathrm{a_n}) - (\mathrm{e_n})$ exist and prove that then the sets

$$\{T_i\}_{0<i\leq n+1}, \{C_\lambda\}_{\lambda\in T_{n+1}}, \{A_\beta^{(i)}\}_{0<i\leq n+1}$$

with the properties $(\mathrm{a}_{n+1}) - (\mathrm{e}_{n+1})$ also exist.

It is easy to see that our assertion is true, when the element x_{n+1} belongs to $\Sigma_{\lambda\in T_n} C_\lambda$ or the group $\Sigma_{\lambda\in T_n} C_\lambda$ coincides with A_β, because in that case it is sufficient to put $T_{n+1} = T_n$ and $A_\beta^{(n+1)} = A_\beta^{(n)}$.

We shall assume now that x_{n+1} is not contained in $\Sigma_{\lambda\in T_n} C_\lambda$. Then the component y_{n+1} of x_{n+1} in the direct summand $A_\beta^{(n)}$ of $(\mathrm{C_n})$ is not the null element. By induction hypothesis (E) is fully consistent with $(\mathrm{C_n})$. Therefore, by (7.2), for the direct summand $A_\beta^{(n)}$ of $(\mathrm{C_n})$ and the element y_{n+1} there exists a finite subset Γ_n of $L(A_\beta^{(n)})$ a set $\{C_\lambda\}_{\lambda\in\Gamma_n}$ of subgroups of G and a group $A_\beta^{(n+1)}$ with the following properties:

$$C_\lambda + \check{E}_\lambda = \bar{E}_\lambda \quad (\lambda \in \Gamma_n), \tag{$\mathrm{a_n^*}$}$$

$$A_\beta^{(n)} = \sum_{\lambda\in\Gamma_n} C_\lambda + A_\beta^{(n+1)}, \text{ where } A_\beta^{(n+1)} = A_\beta^{(n)} \cap G_{L-\Gamma_n}, \tag{$\mathrm{b_n^*}$}$$

$$y_{n+1} \in \sum_{\lambda\in\Gamma_n} C_\lambda, \tag{$\mathrm{c_n^*}$}$$

the following direct decomposition holds $(\mathrm{d_n^*})$

$$G = \sum_{\lambda\in T_n} C_\lambda + \sum_{\lambda\in\Gamma_n} C_\lambda + A_\beta^{(n+1)} + \sum_{\nu\in N-\{\beta\}} A_\nu \tag{$\mathrm{C_{n+1}^*}$}$$

and (E) is fully consistent with $(\mathrm{C_{n+1}^*})$.

Now when we put $T_{n+1} = T_n \cup \Gamma_n$ then we deduce from $(\mathrm{a_n}) - (\mathrm{e_n})$ and $(\mathrm{a_n^*}) - (\mathrm{e_n^*})$ that the sets

$$\{T_i\}_{0<i\leq n+1}, \{C_\lambda\}_{\lambda\in T_{n+1}}, \{A_\beta^{(i)}\}_{0<i\leq n+1}$$

have the properties $(\mathrm{a}_{n+1}) - (\mathrm{e}_{n+1})$.

Thus, we have proved that the sets $\{T_i\}_{0<i<n}$, $\{C_\lambda\}_{\lambda\in T_n}$, $\{A_\beta^{(i)}\}_{0<i\leq n}$ with the properties $(\mathrm{a_n}) - (\mathrm{e_n})$ exist for every natural number n.

We put

$$Q = \bigcup_{n=1,\,2,\,\cdots} T_n. \tag{4}$$

From $(\mathrm{b_n})$, $(\mathrm{e_n})$, and (4) we deduce that

$$[\bigcup_{\lambda\in Q} C_\lambda] = \sum_{\lambda\in Q} C_\lambda.$$

As $(\mathrm{c_n})$ holds for every natural number n, it follows that all elements of (3) belong to $\Sigma_{\lambda\in Q} C_\lambda$, so that $A_\beta \subset \Sigma_{\lambda\in Q} C_\lambda$. But, by $(\mathrm{b_n})$, $C_\lambda \subset A_\beta$; therefore,

$$A_\beta = \sum_{\lambda\in Q} C_\lambda. \tag{5}$$

Further, by $(\mathrm{a_n})$ and (4),

$$C_\lambda + \check{E}_\lambda = \overline{E}_\lambda. \tag{6}$$

Now from (5) and (6) it follows by (7.3) that

$$Q = S(A_\beta,\ \mathrm{E},\ \Omega_{\mathrm{A}}). \tag{7}$$

Comparing (5), (6), and (7) we see that (1) and (2) hold. This completes the proof of the theorem.

§8

In the present section we consider the problem of conditions under which two given direct decompositions of a group have centrally isomorphic refinements. The main result is Theorem (8.2). From this theorem it follows that two given direct decompositions (D) and (A) of a group G have centrally isomorphic refinements if (D) is consistent with (A) and if at least one of the following conditions is satisfied: a) G is countable; b) the number of direct summands in (D) is finite; c) the cardinal number of every direct summand of (A) is not more than countable; d) the partially ordered set $L(\mathrm{D},\ \Omega_{\mathrm{A}})$ satisfies the minimal condition. From Theorem (8.2) it also follows (see Theorems (8.3) and (8.4)) that a direct decomposition (D) of G and *every other* direct decomposition of G have centrally isomorphic refinements if (D) is partially ordered with respect to the set Ω of all normal idempotent endomorphisms of G and if at least one of the following two conditions is satisfied: 1) G is countable; 2) the partially ordered set $L(\mathrm{D},\ \Omega)$ satisfies the minimal condition.

An automorphism ϕ of a group G is called central if for every $x \in G$ the difference $\phi x - x$ belongs to the center of G. An automorphism of a group is central if and only if it is normal.*

Two subgroups of a group G are called centrally isomorphic if there exists a central automorphism of G mapping one of the subgroups onto the other. Below we shall make use of the following proposition:

If the following direct decompositions hold

$$G = A + C = B + C,$$

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* Translator's note: that is, permutable with the inner automorphisms.

*then A and B are centrally isomorphic.**

Two direct decompositions of a group are called *centrally isomorphic* if there exists a central automorphism of the group carrying one of the decompositions into the other. It is not difficult to see that two direct decompositions of a group are centrally isomorphic if a one-to-one correspondence can be set up between their direct summands in which corresponding summands are centrally isomorphic.

(8.1) *Let*

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

be a direct decomposition of a group G, fully consistent with the decomposition

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

in which every direct summand A_ν satisfies at least one of the following two conditions:

1. *A_ν is a finite or countable group;*
2. *the subset $S(A_\nu, \mathrm{E}, \Omega_A)$ of $L(\mathrm{E}, \Omega_A)$ satisfies the minimal condition.*

Then (A) *can be refined to a decomposition centrally isomorphic with* (E).

Proof. By C_λ, $\lambda \in L$, we denote the component of E_λ in the direct summand $A_{\rho(\lambda)}$ of (A). By hypothesis (E) is fully consistent with (A) and every A_ν satisfies at least one of the conditions 1, 2. Therefore, we deduce from (6.6) and (7.4) that the following direct decomposition holds:

$$A_\nu = \sum_{\lambda \in S(A_\nu, \mathrm{E}, \Omega_A)} C_\lambda \qquad (\nu \in N). \tag{1}$$

By (5.5) we have

$$L = \bigcup_{\nu \in N} S(A_\nu, \mathrm{E}, \Omega_A). \tag{2}$$

When we replace in (A) every direct summand A_ν by the formula (1) and take (2) into account, we obtain the direct decomposition:

$$G = \sum_{\lambda \in L} C_\lambda \tag{C}$$

which is, evidently, a refinement of (A).

By (5.11j) we have

$$C_\lambda + \check{E}_\lambda = E_\lambda + \check{E}_\lambda \qquad (\lambda \in L,\ \check{E}_\lambda = U(E_\lambda, \mathrm{E}, \Omega_A)). \tag{3}$$

On the strength of (3) and (E) we can assert that the following direct decomposition holds:

$$G = C_\lambda + \sum_{i \in L-\{\lambda\}} = E_\lambda + \sum_{i \in L-\{\lambda\}} \qquad (\lambda \in L), \tag{4}$$

which shows that for every $\lambda \in L$ the groups C_λ and E_λ are centrally isomorphic; hence, it follows that (E) and (C) are centrally isomorphic. Thus, (C) is the required refinement of (A) centrally isomorphic with (E).

(8.2) **Theorem.** *Let*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

---

*See A. G. Kuroš [5].

be a direct decomposition of a group G, consistent with the decomposition

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

in which every direct summand A_ν satisfies at least one of the two conditions:

1. *A_ν is a finite or countable group;*
2. *the subset $S(A_\nu, \mathrm{D}, \Omega_\mathrm{A})$ of $L(\mathrm{D}, \Omega_\mathrm{A})$ satisfies the minimal condition.*

Then (D) *and* (A) *have centrally isomorphic refinements.*

Proof. By L we denote the set of all ordered pairs (α, ν), $\alpha \in M$, $\nu \in N$, for which the component $E_{(\alpha, \nu)}$ of $A_\nu \cap \overline{D}_\alpha$ in the direct summand D_α of (D) is not the null group. By hypothesis, (D) is consistent with (A). Therefore, by (5.10), the following direct decomposition holds:

$$G = \sum_{\lambda \in L} E_\lambda, \tag{E}$$

and this decomposition is a refinement of (D) and fully consistent with (A).

If the subset $S(A_\nu, \mathrm{D}, \Omega_\mathrm{A})$ of $L(\mathrm{D}, \Omega_\mathrm{A})$ satisfies the minimal condition, then by (5.12) $S(A_\nu, \mathrm{E}, \Omega_\mathrm{A})$ also satisfies the minimal condition. But by hypothesis, every direct summand A_ν of (A) satisfies at least one of the conditions 1, 2. Therefore, every A_ν satisfies at least one of the conditions 1, 2 of (8.1). Therefore, from (8.1) we deduce that there exists a direct decomposition

$$G = \sum_{\lambda \in L} C_\lambda \tag{C}$$

which is a refinement of (A) and centrally isomorphic with (E). Thus, (E) and (C) are the required centrally isomorphic refinements of (D) and (A). This completes the proof of the theorem.

(8.3) Theorem. *A direct decomposition of a countable group G which is partially ordered with respect to the set of all normal idempotent endomorphisms of G and any other decomposition of G have centrally isomorphic refinements.*

Proof. Let G be a countable group and

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

a direct decomposition of G partially ordered with respect to the set Ω of all normal idempotent endomorphisms of G. Further, let

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

be any other direct decomposition of G. The set Ω_A of normal idempotent endomorphisms corresponding to (A) is a subset of Ω. As (D) is partially ordered with respect to Ω and as $\Omega_\mathrm{A} \subset \Omega$, (D) is also partially ordered with respect to Ω_A. Therefore, (D) is consistent with (A). Therefore, we conclude from Theorem (8.2) that (D) and (A) have centrally isomorphic refinements.

(8.4) Theorem. *If a direct decomposition*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

of a group G is partially ordered with respect to the set Ω of all normal idempotent endomorphisms of G, and if $L(D, \Omega_A)$ satisfies the minimal condition, then (D) and any other direct decomposition of G have centrally isomorphic refinements.

Proof. Let

$$G = \sum_{\nu \in N} A_\nu \tag{A}$$

be an arbitrary direct decomposition of G and Ω_A the set of normal idempotent endomorphisms corresponding to (A). As $\Omega_A \subset \Omega$ and, by hypothesis, (D) is partially ordered with respect to Ω, (D) is also partially ordered with respect to Ω_A. Therefore, (D) is consistent with (A). Further, since $\Omega_A \subset \Omega$ and, by hypothesis, the minimal condition holds in $L(D, \Omega)$, it also holds, by (5.12), in $L(D, \Omega_A)$. Thus, every direct summand A_ν of (A) satisfies condition 2 of Theorem (8.2). Therefore, by Theorem (8.2), (D) and (A) have centrally isomorphic refinements. This completes the proof.

§9

In this section we shall derive necessary and sufficient conditions under which a given direct decomposition of a group is partially ordered with respect to the set of *all* normal endomorphisms of the group (Theorem (9.5)). In connection with this we shall obtain new conditions under which a given direct decomposition and any other direct decomposition of the group have centrally isomorphic refinements (Theorem (9.6) and (9.7)).

A homomorphism of a group H into a group B will be called non-trival if the image of H under this homomorphism is not the null group.

In this section we shall make use of the following theorem which is due to Fitting [8]:

(9.1) *If $G = B + H$, then H is an admissible subgroup with respect to the set of all normal endomorphisms of G if and only if there exists no non-trivial homomorphism of H into the center of G.*

We shall prove the following proposition which generalizes Theorem (9.1).

(9.2) *If $G = \Sigma_{\alpha \in M} D_\alpha$ and Γ is a subset of the index set M, then the subgroup $H = \Sigma_{\beta \in \Gamma} D_\beta$ if admissible with respect to the set of all normal endomorphisms of G if and only if for $i \in \Gamma$ and $k \in M - \Gamma$ there exists no non-trivial homomorphism of D_i into the center of D_k.*

Proof. By ϕ_α, $\alpha \in M$, we denote the idempotent endomorphism corresponding to the direct summand D_α of the decomposition

$$G = \sum_{\alpha \in M} D_\alpha. \tag{D}$$

We put

$$B = \sum_{\gamma \in M - \Gamma} D_\gamma. \tag{1}$$

Then clearly

$$G = B + H. \tag{2}$$

1. We first prove the sufficiency of the condition. Let us assume that there

exists no non-trivial homomorphism of D_i into the center of D_k if $i \in \Gamma$, $k \in M-\Gamma$. We shall prove that H is then an admissible subgroup with respect to the set of all normal endomorphisms of G. By (9.1) it is sufficient to prove for this purpose that there exists a non-trival homomorphism of H into the center of B.

Suppose there exists a non-trivial homomorphism ψ of H into the center of B. If we assume that $\psi D_\beta = \{0\}$ for every $\beta \in \Gamma$, then we have $\psi H = \{0\}$, because $H = \Sigma_{\beta \in \Gamma} D_\beta$; but this is impossible since ψ is a non-trivial homomorphism. Therefore, there exists an index $i \in \Gamma$ such that

$$\psi D_i = \{0\}. \tag{3}$$

Hence, taking (1) into account we conclude that there exists at least one index $k \in M - \Gamma$ such that the component of ψD_i in the direct summand D_k of (D) is different from zero; that is,

$$\phi_k \psi D_i \neq \{0\}. \tag{4}$$

It is easy to see that the homomorphism $\phi_k\psi$ maps D_i into the center of D_k. This follows from the fact that ψH belongs to the center of B and that $D_i \subset H$. Therefore, taking (4) into account, we deduce that $\phi_k\psi$ induces a non-trivial homomorphism of D_i into the center of D_k, where $i \in \Gamma$ and $k \in M - \Gamma$. By hypothesis, however, this is impossible. Therefore, the assumption on the existence of a non-trivial homomorphism of H into the center of B must be rejected, and by (9.1) H is an admissible subgroup with respect to the set of all normal endomorphisms of G.

2. We now prove the necessity of the condition. Let us assume that H is admissible with respect to all normal endomorphisms of G. Then there exists no non-trivial homomorphism of D_i into the center of D_k, provided $i \in \Gamma$, $k \in M - \Gamma$. For if a non-trivial homomorphism η of D_i into the center of D_k existed, then the homomorphism $\eta\phi_i$ would induce a non-trivial homomorphism of H into the center of B, which is impossible, by (9.1).

(9.3) *Let*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

be a direct decomposition of a group G, partially ordered with respect to the set Ω of all normal endomorphisms of G. Then there exists no non-trivial homomorphism of D_i into the center of D_k, $i, k \in M$, whenever the indices i, k regarded as elements of $L(\mathrm{D}, \Omega)$ satisfy the condition: $i < k$ or i is incomparable with k.

Proof. Let D_i and D_k be two distinct direct summands of (D) and suppose that $i < k$ or i is incomparable with k when i and k are regarded as elements of $L(\mathrm{D}, \Omega)$. By V_i we denote the combinatorial closure of i in $L(\mathrm{D}, \Omega)$. Then, obviously,

$$i \in V_i \tag{1}$$

$$k \notin V_i. \tag{2}$$

By (2.9)

$$\overline{D}_i = \sum_{\alpha \in V_i} D_\alpha, \tag{3}$$

where $\overline{D}_i = V(D_i, \mathrm{D}, \Omega)$. From the definition of $\overline{D}_i$ it follows that $\overline{D}_i$ is an Ω-subgroup of G. Hence, taking (1), (2), and (3) into account, we deduce from (9.2) that there exists no non-trivial homomorphism of D_i into the center of D_k.

(9.4) *If a direct decomposition*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

of a group G contains no null summands and if the index set M is partially ordered such that whenever $i < k$, $i, k \in M$, there exists no non-trivial homomorphism of D_i into the center of D_k, then (D) *is partially ordered with respect to the set of all normal endomorphisms of G.*

Proof. By Ω we denote the set of all normal endomorphisms of G. Let us assume that an order relation is introduced into the index set M which turns it into a partially ordered set θ satisfying the following condition:

a) if $i < k$ or i is incomparable with k, $i, k \in \theta$, then there exists no non-trivial homomorphism of D_i into the center of D_k.

Let D_α and D_β be two distinct direct summands of (D). We shall show that among the groups belonging to $\Delta(\mathrm{D}, \Omega)$ there exists one that contains one of the direct summands D_α, D_β, but not the other. This will prove, by (1.4b), that (D) is partially ordered with respect to Ω. It is easy to see that at least one of the following relation holds: $\alpha \notin V_\beta$, $\beta \notin V_\alpha$, where V_α and V_β are the combinatorial closures of α and β in θ. Let us assume, for example, that

$$\alpha \notin V_\beta.$$

If $i \notin V_\beta$ and $k \in \theta - V_\beta$, then, obviously, $i < k$ or i is incomparable with k. As θ satisfies a), it follows, hence, that there esists no non-trivial homomorphism of D_i into the center of D_k. Therefore, we conclude, by (9.2), that

$$H_\beta = \sum_{i \in V_\beta} D_i$$

is an Ω-subgroup of G. Therefore,

$$H_\beta \in \Delta(\mathrm{D}, \Omega).$$

Further, since by hypothesis (D) contains no null summands, we deduce from (1), (2), and the obvious relation $\beta \in V_\beta$ that H_β contains D_β, but not D_α. Thus, we have proved that (D) is partially ordered with respect to the set Ω of all normal endomorphisms of G.

(9.5) Theorem. *In order that a direct decomposition*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

of a group G be partially ordered with respect to the set of all normal endomorphisms of G it is necessary and sufficient that (D) *contains no null summands and that the index set M can be partially ordered such that, whenever $i < k$ or i is incomparable with k, $i, k \in M$, there exists no non-trivial homomorphism of D_i into the center of D_k.*

This theorem follows immediately from propositions (9.3) and (9.4).

(9.6) Theorem. *Let*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

be a direct decomposition of a countable group G. *If the index set* M *can be partially ordered such that there exists no non-trivial homomorphism of* D_i *into the center of* D_k, $i, k \in M$, *whenever* $i < k$ *or* i *is incomparable with* k, *then* (D) *and any other direct decomposition of* G *have centrally isomorphic refinements.*

Proof. Let us assume that the condition of the theorem are satisfied. By (9.5), (D) is then partially ordered with respect to the set of all normal endomorphisms of G. Therefore, by (4.3), (D) is also partially ordered with respect to the set of all normal idempotent endomorphisms of G. Hence, as G is a countable group, it follows by (8.3) that (D) and any other direct decomposition of G have centrally isomorphic refinements.

(9.7) Theorem. *Let*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

be a direct decomposition of a group G. *If the index set* M *can be partially ordered such that* a) *there exists no non-trivial homomorphism of* D_i *into the center of* D_k, $i, k \in M$, *whenever* $i < k$ *or* i *is incomparable with* k, *and that* b) *the partially ordered set into which* M *is turned satisfies the minimal condition, then* (D) *and any other direct decomposition of* G *have centrally isomorphic refinements.*

Proof. Let us assume that an order relation is introduced into the index set which turns it into a partially ordered set θ satisfying the following two conditions:

(a) if $i < k$ or i is incomparable with k, then there exists no non-trivial homomorphism of D_i into the center of D_k;

(b) the partially ordered set θ satisfies the minimal condition.

By (9.4) (D) is partially ordered with respect to the set of all normal endomorphisms of G. Therefore, by (4.3), (D) is also partially ordered with respect to the set Ω of all normal idempotent endomorphisms of G. We shall show that θ satisfies the following condition:

(c) if β is an arbitrary element of θ, then the combinatorial closure V_β^* of β in $L(\mathrm{D}, \Omega)$ is a subset of the combinatorial closure V_β of β in θ.

We put

$$H_\beta = \sum_{i \in V_\beta} D_i \qquad (\beta \in \theta).$$

As θ satisfies the condition (a), we find, exactly as in the proof of (9.4), that H_β is admissible with respect to the set of all normal endomorphisms of G. Therefore, the group $\Sigma_{i \in V_\beta} D_i$ is admissible with respect to Ω and so, by (2.5), V_β is closed in $L(\mathrm{D}, \Omega)$. Hence, $V_\beta^* \subset V_\beta$, because $\beta \in V_\beta$ and V_β^* is the least closed subset of $L(\mathrm{D}, \Omega)$ containing β. Thus, θ satisfies condition (c).

Since θ satisfies conditions (b) and (c) it is not difficult to see that $L(\mathrm{D},\Omega)$ satisfies the minimal condition. Thus, (D) satisfies the conditions of Theorem (8.4) and, therefore, (D) and any other direct decomposition of G have centrally isomorphic refinements. This completes the proof.

It is easy to see that the following are special cases of Theorem (9.7):

(a) a theorem of Golovin [1].

Two direct decompositions of an arbitrary group for which the center remains both times entirely in one of the direct factors always have isomorphic refinements;

(b) a generalization of Golovin's theorem given by Kuroš [4]:

If two direct decompositions of an arbitrary group are given such that in at least one of them the center of the group remains entirely in one of the direct factors, then the two decompositions have centrally isomorphic refinements.

§10

In the present section we consider the problem of conditions under which two arbitrary direct decompositions of a group have centrally isomorphic refinements. The main result is Theorem (10.3).

(10.1) *If any two direct decompositions of a group* C *have centrally isomorphic refinements, if* H *is a subgroup of* C *admissible with respect to the set of all normal idempotent endomorphisms of* C, *and if*

$$C = A + H, \tag{1}$$

then any two direct decompositions of H *or of* A *have centrally isomorphic refinements.*

Proof. Let us assume that any two direct decompositions of C have centrally isomorphic refinements. Let us assume, further, that H is admissible with respect to the set of all normal idempotent endomorphisms of C and that (1) holds. Let

$$A = \sum_{\alpha \in N} B_\alpha \tag{B}$$

$$A = \sum_{\beta \in M} F_\beta \tag{F}$$

be two direct decompositions of A and

$$H = \sum_{l \in Q} H_l \tag{H}$$

$$H = \sum_{k \in R} E_k \tag{E}$$

two direct decompositions of H. We shall show that (B) and (F) as well as (H) and (E) have centrally isomorphic refinements.

From (1), (B), (F), (H), (E) we deduce that C has the following direct decompositions:

$$C = \sum_{\alpha \in N} B_\alpha + \sum_{l \in Q} H_l, \tag{2}$$

$$C = \sum_{\beta \in M} F_\beta + \sum_{k \in R} E_k. \tag{3}$$

By hypothesis, any two direct decompositions of C have centrally isomorphic refinements, in particular (2) and (3).

Let

$$C = \sum_{i \in T} C_i \tag{C}$$

$$C = \sum_{i \in T} Z_i \tag{Z}$$

be centrally isomorphic refinements of (2) and (3), respectively. Then there exists a normal automorphism of ϕ of C carrying (C) into (Z), and we can assume that ϕ maps C_i onto Z_i,

$$\phi C_i = Z_i \qquad (i \in T). \tag{4}$$

Now (C) is a refinement of (1), because (C) is a refinement of (2), and (2) a refinement of (1). Therefore, there exists a subset Γ of T such that

$$H = \sum_{i \in \Gamma} C_i \tag{5}$$

and

$$A = \sum_{i \in T-\Gamma} C_i. \tag{6}$$

Now, as (C) is a refinement of (2), it is not difficult to see that (5) and (6) are refinements of (H) and (B), respectively.

By hypothesis, H is admissible with respect to the set of all normal idempotent endomorphisms of C. Hence, it follows by (4.3) that H is admissible with respect to the set of all normal automorphisms of C. Therefore,

$$\phi H = H. \tag{7}$$

From (4), (5), (7) we deduce that

$$H = \sum_{i \in \Gamma} \phi C_i = \sum_{i \in \Gamma} Z_i. \tag{8}$$

Now (Z) is a refinement of (1), because (Z) is a refinement of (3), and (3) of (1). Hence, it follows from (8) that

$$A = \sum_{i \in T-\Gamma} Z_i. \tag{9}$$

As (Z) is a refinement of (3), it is not difficult to see that (8) and (9) are refinements of (E) and (F), respectively.

We have proved above that (5) and (8) are refinements of (H) and (E), and (6) and (9) of (B) and (F), respectively. Further, since ϕ is a normal automorphism of C mapping C_i onto Z_i, $i \in T$, C_i and Z_i are centrally isomorphic . Therefore, (5) and (8) are the required centrally isomorphic refinements of (H) and (E), and (6) and (9) of (B) and (F), respectively.

(10.2) *Let*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

be a direct decomposition of a grroup G, partially ordered with respect to the set Ω of all normal idempotent endomorphisms of G. In order that any two direct decompositions of G have centrally isomorphic refinements it is necessary that for every $\alpha \in M$ any two direct decompositions of D_α have centrally isomorphic refinements.

Proof. Let us assume that any two direct decompositions of G have centrally isomorphic refinements. We shall show that then for every $\alpha \in M$ any two direct decompositions of D_α have centrally isomorphic refinements. By V_α we denote the combinatorial closure of α in $L(\mathrm{D}, \Omega)$ and we put

$$\overline{D}_\alpha = \sum_{i \in V_\alpha} D_i,$$

$$\check{D}_\alpha = \sum_{i \in V_\alpha - \{\alpha\}} D_i.$$

Then

$$\overline{D}_\alpha = D_\alpha + \check{D}_\alpha \quad (\alpha \in M). \tag{1}$$

As V_α and $V_\alpha - \{\alpha\}$ are closed subsets of $L(\mathrm{D}, \Omega)$, $\overline{D}_\alpha$ and $\check{D}_\alpha$ are, by (2.5), Ω-subgroups of G.

By hypothesis any two direct decompositions of G have centrally isomorphic refinements. Moreover, $\overline{D}_\alpha$ is a direct summand of G and is admissible with respect to the set Ω of all normal idempotent endomorphisms of G. Therefore, we deduce from (10.1) that any two direct decompositions of $\overline{D}_\alpha$ have centrally isomorphic refinements. Further, the subgroup $\check{D}_\alpha$ of $\overline{D}_\alpha$ is admissible with respect to the set of all normal idempotent endomorphisms of $\overline{D}_\alpha$. This follows from (4.2), because $\check{D}_\alpha$ is admissible with respect to Ω and $\overline{D}_\alpha$ is a direct summand of G. Moreover, $\overline{D}_\alpha = D_\alpha + \check{D}_\alpha$ and any two direct decompositions of $\overline{D}_\alpha$ have centrally isomorphic refinements. Therefore, we deduce from (10.1) that any two direct decompositions of D_α, $\alpha \in M$, have centrally isomorphic refinements.

(10.3) Theorem. *Let*

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

be a direct decomposition of a group G, partially ordered with respect to the set Ω of all normal idempotent endomorphisms of G. We assume that at least one of the following two conditions is satisfied:

1. *G is a finite or countable group;*
2. *the partially ordered set $L(\mathrm{D}, \Omega)$ satisfies the minimal condition.*

In order that any two direct decompositions of G have centrally isomorphic refinements it is necessary and sufficient that for every $\alpha \in M$ any two direct decompositions of D_α have centrally isomorphic refinements.

Proof. By (10.2) the conditions of the theorem are necessary; let us show that they are sufficient. We assume that for every $\alpha \in M$ any two direct decompositions of D_α have centrally isomorphic refinements. Let

$$G = \sum_{\nu \in N} A_\nu, \tag{A}$$

$$G = \sum_{k \in \Gamma} B_k \tag{B}$$

be two direct decompositions of G. We shall show that (A) and (B) have centrally isomorphic refinements. By hypothesis (D) is partially ordered with respect to Ω and satisfies at least one of the conditions 1, 2. Therefore, we deduce from Theorem (8.3) and (8.4) that (D) and (A) on the one hand, (D) and (B) on the other hand, have centrally isomorphic refinements. Let

$$G = \sum_{\lambda \in L} E_\lambda \tag{E}$$

and

$$G = \sum_{\lambda \in L} C_\lambda \tag{C}$$

be centrally isomorphic refinements of (D) and (A), respectively, and ϕ a normal automorphism of G carrying (E) into (C). Further, let

$$G = \sum_{n \in P} F_n \tag{F}$$

and

$$G = \sum_{n \in P} Z_n \tag{Z}$$

be centrally isomorphic refinements of (D) and (B), respectively, and ψ a normal automorphism of G carrying (F) into (Z).

As (E) and (F) are refinements of (D), there exist subsets L_α and P_α of L and P, respectively, such that

$$D_\alpha = \sum_{\lambda \in L_\alpha} E_\lambda \quad (\alpha \in M), \tag{E$_\alpha$}$$

$$D_\alpha = \sum_{n \in P_\alpha} F_n \quad (\alpha \in M). \tag{F$_\alpha$}$$

By hypothesis, for every $\alpha \in M$ any two direct decompositions of D_α have centrally isomorphic refinements, in particular (E$_\alpha$) and (F$_\alpha$). Let

$$D_\alpha = \sum_{i \in T_\alpha} H_i \quad (\alpha \in M) \tag{H$_\alpha$}$$

and

$$D_\alpha = \sum_{i \in T_\alpha} R_i \quad (\alpha \in M) \tag{R$_\alpha$}$$

be centrally isomorphic refinements of (E$_\alpha$) and (F$_\alpha$), $\alpha \in M$, respectively. When we replace every direct summand D_α in (D) by the expressions (H$_\alpha$) and (R$_\alpha$), respectively, and when we put $T = \bigcup_{\alpha \in M} T_\alpha$, we obtain the direct decompositions:

$$G = \sum_{i \in T} H_i, \tag{H}$$

$$G = \sum_{i \in T} R_i \tag{R}$$

which are easily seen to be refinements of (E) and (F), respectively. Since for every $\alpha \in M$, (H$_\alpha$) and (R$_\alpha$) are centrally isomorphic, (H) and (R) are also centrally isomorphic. Therefore, there exists a normal automorphism η of G carrying (H) into (R).

We put

$$X_i = \phi^{-1} H_i \quad (i \in T), \tag{1}$$

$$Y_i = \psi R_i \quad (i \in T). \tag{2}$$

It is easy to see that the automorphisms ϕ^{-1} and ψ of G carry (H) and (R) respectively, into

$$G = \sum_{i \in T} X_i, \tag{X}$$

$$G = \sum_{i \in T} Y_i. \tag{Y}$$

As (H) is a refinement of (E) and ϕ^{-1} carries (E) and (H) into (C) and (X), respectively, (X) is a refinement of (C). Similarly (Y) is a refinement of (Z), as (R) is a refinement of (F) and ψ carries (F) and (R) into (2) and (Y), respectively. It is easy to see that the automorphism θ,

$$\theta = \psi \eta \phi,$$

carries (X) into (Y). Moreover, θ being the product of the normal automorphisms ψ, η, ϕ of G is a normal automorphism of G. Finally, (X) and (Y) are refinements of (A) and (B), respectively, because (C) and (Z) are refinements of (A) and (B), and (X) and (Y) refinements of (C) and (Z). Thus, (X) and (Y) are the required centrally isomorphic refinements of (A) and (B), and the normal automorphism θ carries (X) into (Y). This completes the proof of the theorem.

(10.4) *Suppose that* G *has a direct decomposition*

$$G = H + F, \tag{1}$$

in which the direct summand H *contains no elements of finite order, other than null, while every element of* F *has finite order. In order that any two direct decompositions of* G *have centrally isomorphic refinements it is necessary and sufficient that any two direct decompositions of* F *as well as of* H *have centrally isomorphic refinements.*

Proof. It is easy to see that F is admissible with respect to the set of all normal endomorphisms of G, in particular, of all normal idempotent endomorphisms of G. Therefore, by (4.3), (1) is partially ordered with respect to the set of all normal idempotent endomorphisms of G. Therefore, Theorem (10.4) is a special case of Theorem (10.3).

§11

The object of the present section is the proof of a Theorem (11.2) due to Baer [6] which will be required in the next section. The proof we shall give below is different from Baer's proof.

In the proof of proposition (11.1) we shall make use of the concept of the height of an element which is defined as follows. Let B be an abelian torsion-free group and x and element of B. The *height* of x in B shall be the set of all rational integers n with $x \in nB$. We shall denote the height of x in B by the symbol $h(x, B)$. The relation $h(x, B) \geq h(y, B)$ shall mean that $h(y, B)$ is a subset of $h(x, B)$. The relation $h(x, B) > h(y, B)$ shall mean that $h(y, B)$ is a proper subset of $h(x, B)$. Further, we shall say the height $h(x, B)$ of x is incomparable with the height $h(y, B)$ of y if neither set $h(x, B)$, $h(y, B)$ is contained in the other.

It is easy to verify that for every integer r and every element $x \in B$ we have $h(rx, B) = rh(x, B)$.

(11.1) *Let* B *be an abelian torsion-free group,* ψ *a homomorphism of* B *into a torsion-free group* T *of rank 1 and* A *the kernel of this homomorphism. Suppose,* *further, that every serving subgroup of B of rank 1, not contained in A, is isomorphic to T. Then A is a direct summand of B.*

Proof. 1. Let b be an element of B, not contained in A. We shall show that the coset $b + A$ contains an element c of maximal height; that is, an element such that for every element $x \in b + A$ either $h(c, B) \geq h(x, B)$ or $h(c, B)$ is incomparable with $h(x, B)$. Suppose that such a maximal element does not exist. Then there is an infinite sequence $x_1, x_2, \cdots, x_n, \cdots$ in the coset $b + A$ such that

$$h(x_1, B) < h(x_2, B) < \cdots < h(x_n, B) < \cdots. \tag{1}$$

By B_i we denote the serving subgroup of B of rank 1, containing x_i, and we put $T_i = \psi B_i$, $i = 1, 2, \cdots$. Since $h(x_i, B) < h(x_k, B)$ for $i < k$, and x_i, x_k belong to the same coset $b + A$, and A is the kernel of the homomorphism ψ, it is not difficult to see that the following relations hold:

$$T_i \subset T_k,\ T_i \neq T_k \qquad (i<k,\ \ i, k = 1, 2, \cdots). \tag{2}$$

By hypothesis every serving subgroup of B, of rank 1, not contained in A, is isomorphic to T, so that

$$B_i \cong T \qquad (i = 1, 2, \cdots). \tag{3}$$

As $x_i \notin A$ and B_i a serving subgroup of B containing x_i, we have $B_i \cap A = \{0\}$. Since A is the kernel of ψ, it follows, hence, that $\psi B_i \cong B_i$ so that we obtain from (3) that

$$T_i \cong T \qquad (i = 1, 2, \cdots). \tag{4}$$

The relations (2) and (4) show that there exists an *infinite ascending* sequence of subgroups of T, isomorphic to T itself. But this is impossible because T is a group of rank 1. This proves that every coset $b + A$, where $b \in B$, $b \notin A$, contains at least one element of maximal height.

2. Let b be an element of B, not belonging to A, and c an element of $b + A$, having maximal height in B among the elements of this coset. We shall show that

$$h(c, B) \geq h(z, B) \tag{5}$$

for every $z \in b + A$. By C and Z we denote the serving subgroups of B, of rank 1, containing c and z respectively. By hypothesis C and Z are isomorphic. Let ϕ be an isomorphism mapping C onto Z. Then, as Z is a group of rank 1, there exist integers r and s such that

$$z = (r/s)\, \phi c, \tag{6}$$

and we can assume that r and s are co-prime and $r > 0$. Clearly (6) implies

$$\phi(rc) = sz. \tag{7}$$

As C and Z are serving subgroups of B and ϕ an isomorphic mapping of C onto Z, we deduce from (7) that

$$h(rc, B) = h(sz, B). \tag{8}$$

Since r and s are co-prime, there exist integers l and k such that

$$lr + ks = 1. \tag{9}$$

We put

$$g = lrc + ksz. \tag{10}$$

Let $\overline{g} = g + A$. As $c, z \in \overline{b} = b + A$, we have by (10)

$$\overline{g} = lr\overline{b} + ks\overline{b} = (lr + ks)\,\overline{b},$$

hence, by (9) $\overline{g} = \overline{b}$; and therefore,

$$g \in \overline{b} = b + A. \tag{11}$$

From (8) and (10) we deduce that

$$h(g, B) \geq h(rc, B);$$

hence,

$$h(g, B) \geq rh(c, B). \tag{12}$$

On the other hand, since by (11) $g \in b + A$ and c has maximal height in B among the elements of this coset, we have

$$h(g, B) \leq h(c, B). \tag{13}$$

As r is a positive integer, it follows from (12) and (13) that $r = 1$. Hence, by (8), (5) holds.

3. Let b be an element of B, not belonging to A, c an element of $b + A$ having maximal height in B among the elements of this coset, and C the serving subgroup of B, of rank 1, that contains c. We shall show that

$$B = A + C. \tag{14}$$

As C is a serving subgroup of B, we have

$$h(c, C) = h(c, B). \tag{15}$$

By $\overline{B}$ we denote the factor-group $B - A$,

$$\overline{B} = B - A, \tag{16}$$

and by $\overline{C}$ the image of C under the natural homomorphism of B onto $\overline{B}$. We put $\overline{c} = c + A = b + A$. As $\overline{c}$ is the image of c under the natural homomorphism of B onto $\overline{B}$, it is easy to see that

$$h(\overline{c}, \overline{C}) \geq h(c, C). \tag{17}$$

From (5) we conclude that

$$h(c, B) \geq h(\overline{c}, \overline{B}). \tag{18}$$

Now comparing (15), (17), and (18) we obtain

$$h(\overline{c}, \overline{C}) \geq h(\overline{c}, \overline{B});$$

hence, as $\overline{C}$ is a subgroup of $\overline{B}$,

$$h(\overline{c}, \overline{C}) = h(\overline{c}, \overline{B}). \tag{19}$$

By hypothesis $\overline{B}$ is a group of rank 1 ($\overline{B} \cong T$); moreover, $\overline{C}$ is a subgroup of $\overline{B}$, and (19) shows that $\overline{c}$ has one and the same height in $\overline{C}$ and $\overline{B}$. As $\overline{c}$ is not the null element of $\overline{C}$, it follows that $\overline{C}$ coincides with $\overline{B}$

$$\overline{C} = \overline{B}. \tag{20}$$

Now the complete inverse image of $\overline{C}$ under the natural homomorphism of B onto

$\overline{B}$ is the group $[A, C]$ so that it follows from (20) that

$$B = [A, C]. \tag{21}$$

By (16) the kernel of the natural homomorphism of B onto $\overline{B}$ is A. Further, the image $\overline{C}$ of C under this homomorphism contains no element of finite order, except the null element, but is not the null group because $\overline{C} = \overline{B} \cong T$. Therefore, we can assert that

$$A \cap C = \{0\}. \tag{22}$$

Now from (21) and (22) we deduce that (14) holds and so A is a direct summand of B.

(11.2) Theorem. *If an abelian torsion-free group decomposes into the direct sum of subgroups isomorphic to one and the same group* T *of rank* 1, *then every direct summand of the group also decomposes into the direct sum of subgroups isomorphic to* T.

Proof. Let G be an abelian torsion-free group and

$$G = \sum_{0 \leq a < \gamma} G_a \tag{1}$$

a decomposition of G into the direct sum of subgroups isomorphic to a group T of rank 1. We assume that the index a in (1) ranges over the set of all ordinal numbers less than γ. Let C be a non-null group that is a direct summand of G. We shall show that C decomposes into the direct sum of subgroups isomorphic to T. By C_0 we denote the null subgroup of C and we put

$$C_i = C \cap \sum_{0 \leq a < i} G_a \quad (0 < i \leq \gamma). \tag{2}$$

It is easy to see that

$$C = \bigcup_{0 \leq i \leq \gamma} C_i. \tag{3}$$

It is not difficult to verify that the intersection of any two serving subgroups of an abelian torsion-free group is also a serving subgroup. But direct summands of a group are serving subgroups of it. Hence, we deduce, taking (2) into account, that C_i, $i \leq \gamma$, is a serving subgroup of G.

We shall show now that the factor-group $C_{i+1} - C_i$ is isomorphic to T, whenever $C_{i+1} \neq C_i$,

$$C_{i+1} - C_i \cong T \quad (0 \leq i < \gamma,\ C_{i+1} \neq C_i). \tag{4}$$

By ϕ_i we denote the homomorphism of C_{i+1} into G_i that associates with every element $c \in C_{i+1}$ its component in the direct summand G_i of (1). The kernel of ϕ_i is, obviously, C_i and the image of C_{i+1} under this homomorphism is the component S_i of C_{i+1} in the direct summand G_i of (1). Therefore,

$$C_{i+1} - C_i \cong S_i \quad (0 \leq i < \gamma). \tag{5}$$

We assume that $C_{i+1} \neq C_i$. Then there exists an element $c \in C_{i+1}$ having a non-null component in G_i. By Z we denote the serving subgroup of G, of rank 1, that contains c. As G is torsion-free, $c \in C_{i+1}$, and C_{i+1} a serving subgroup of G, we have $Z \subset C_{i+1}$. The component Z^* of Z in G_i is isomorphic to Z, because

G_i is torsion-free, Z a group of rank 1, and Z^* not the null group. By hypothesis G is the direct sum of groups isomorphic to T. Therefore, every serving subgroup of G, of rank 1, is isomorphic to T. Therefore, $Z \cong T$. Hence, it follows, as $Z^* \cong Z$, that

$$Z^* \cong T. \tag{6}$$

Furthermore, since $Z \subset C_{i+1}$,

$$Z^* \subset S_i \subset G_i. \tag{7}$$

Moreover, by hypothesis,

$$G_i \cong T. \tag{8}$$

As T is a group of rank 1, we deduce from (6), (7), and (8) that S_i is isomorphic to T, and thus, (4) holds by (5).

We have mentioned above that every serving subgroup of G, of rank 1, is isomorphic to T. As C_{i+1} is a serving subgroup of G, every serving subgroup of C_{i+1} is also a serving subgroup of G. Therefore, every serving subgroup of C_{i+1}, of rank 1, is isomorphic to T. Moreover, by (4) there exists a homomorphism of C_{i+1} onto T with kernel C_i. Hence, it follows by (11.1), that C_i is a direct summand of C_{i+1}, in other words, that there exists a subgroup T of C_{i+1} such that

$$C_{i+1} = C_i + T_i \qquad (0 \leq i < \gamma,\ C_{i+1} \neq C_i). \tag{9}$$

Now from (3), (9), and (10) we deduce that C decomposes into the direct sum of the subgroups T_i, isomorphic to T, $C = \sum_{\substack{0 \leq i < \gamma \\ C_{i+1} \neq C_i}} T_i$. This completes the proof.

§12

(12.1) Definition. A group is called *completely decomposable* if it is the direct sum of subgroups of rank 1.*

This section is devoted to a solution of the following two problems:

Do any two direct decompositions of a completely decomposable group possess isomorphic refinements?

Is every direct sum of completely decomposable groups also completely decomposable?

These problems are closely connected with one another and it is easy to verify that the solution of one leads to a solution of the other.

In the present section we shall present solutions of these problems for countable groups, for periodic groups, and for one class of non-countable mixed groups.

(12.2) *Let H be a mixed abelian group whose maximal periodic subgroup F is a direct summand. In order that any two direct decompositions of H have isomorphic refinements it is necessary and sufficient that F and the factor-group $H - F$ have the same property.*

This proposition follows immediately from Theorem (10.4).

(12.3) Theorem. *Any two direct decompositions of a completely decomposable*

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* We recall that a group is said to be of rank 1 if an arbitrary finite set of its elements generates a cyclic subgroup.

periodic group have isomorphic refinements.

Proof. Let G be a periodic completely decomposable group. Then it has a decomposition into the direct sum of indecomposable subgroups of rank 1.

Let

$$G = \sum_{i \in N} C_i \tag{C}$$

be such a decomposition. It is well known (see Prüfer [7]) that an indecomposable periodic group of rank 1 is either a cyclic primary group or a quasi-cyclic primary group (a group of type p^{∞}). Let $\{C_i\}_{i \in N_1}$ be the set of all cyclic direct summands of (C) and $\{C_i\}_{i \in N_2}$ the set of all quasi-cyclic direct summands of (C). We put

$$Z = \sum_{i \in N_1} C_i$$

and

$$P = \sum_{i \in N_2} C_i .$$

Then, evidently,

$$G = Z + P, \tag{1}$$

and P is the maximal complete subgroup of G, while Z is a direct sum of cyclic groups. P is also the maximal complete subgroup of G, and, therefore, a fully invariant subgroup of G. Therefore, (1) is partially ordered with respect to the set of all idempotent endomorphisms of G. It is known that any two direct decompositions of a complete group can be refined to decompositions into direct sums of quasi-cyclic subgroups and that any two decompositions of a complete periodic group into direct sums of quasi-cyclic subgroups are isomorphic. Moreover, any two direct decompositions of a group that is a direct sum of cyclic subgroups have isomorphic refinements (see Kulikov [2]). Thus, any two decompositions both of the direct summand Z and the direct summand P in (1) have isomorphic refinements. Therefore, by (10.3), we deduce that any two direct decompositions of G have isomorphic refinements.

(12.4) **Definition.** A direct decomposition of an abelian group G,

$$G = \sum_{\alpha \in M} D_\alpha \tag{D}$$

is called a *canonical decomposition* of G if the following two conditions are satisfied:

(a) any two non-null serving subgroups of G, of rank 1, that belong to one and the same direct summand of (D) are isomorphic;

(b) any two non-null serving subgroups of G, of rank 1, that belong to distinct direct summands of G, are not isomorphic.

It is clear that not every abelian group possesses canonical decompositions, but that, for example, completely decomposable groups possess canonical decompositions.

It is easy to see that to every decomposition of a completely decomposable group into the direct sum of indecomposable groups of rank 1 there corresponds in a natural manner a canonical decomposition of the group. For let G be a completely decomposable group and

$$G = \sum_{i \in N} C_i \tag{C}$$

a decomposition of G into the direct sum of *indecomposable* subgroups of rank *1*. With (C) we associate the direct decomposition

$$G = \sum_{a \in M} D_a, \tag{D}$$

which is obtained from (C) by combining into a single direct summand all isomorphic direct summands of (C). It is not difficult to see that (D) satisfies the conditions (a) and (b) of (12.4) so that it is a canonical decomposition of G. Here (C) is, obviously, a refinement of (D). We shall call (D) the canonical decomposition of G *corresponding to* (C).

(12.5) *Every canonical decomposition of an abelian torsion-free group is partially ordered with respect to the set of all endomorphisms of the group.*

Proof. Let G be an abelian torsion-free group and

$$G = \sum_{a \in M} D_a \tag{D}$$

a canonical decompostiion of it. We shall show that (D) is partially ordered with respect to the set of all endomorphisms of G.

With every direct summand D_a of (D) we associate an abelian torsion-free group R_a, of rank *1*, with the property that every non-null serving subgroup of D_a, of rank *1*, is isomorphic with R_a. Such a group R_a exists and is unique, up to isomorphism, because (D), as a canonical decomposition, satisfies condition (a) of (12.4).

We introduce an order relation into the index set M in the following way: if i and k are two distinct elements of M, then we put $i < k$ if and only if there exists an isomorphic mapping of R_k into R_i, but no isomorphic mapping of R_i into R_k. As R_a is a torsion-free group of rank *1* for every $a \in M$, it is easy to see that this order relation turns M into a partially ordered set. We mention that two elements i and k of this partially ordered set are incomparable if and only if neither group R_i, R_k can be mapped isomorphically into the other.

Let D_i, D_k be two direct summands of (D) whose indices satisfy the condition $i < k$. We shall show that there exists no non-trivial homomorphism of D_i into D_k. Suppose that such a homomorphism ψ exists. Then there is an element x for which $\psi x \neq 0$. By C we denote the serving subgroup of D_i, of rank *1*, that contains x, and by Z the serving subgroup of D_k, of rank *1*, that contains ψx. Then

$$C \cong R_i, \tag{1}$$

$$Z \cong R_k, \tag{2}$$

both C and Z are serving subgroups of G, because D_i and D_k are direct summands of G, and C and Z are not the null groups, since x and ψx are non-null elements of C and Z, respectively. As C is a torsion-free group of rank *1* and x a non-null element of C, every element of C can be represented in the form $(r/s)x$, where r and s are rational integers, Since ψ is a homomorphism of D_i into D_k and D_i, D_k are abelian torsion-free groups, we have

$$\psi((r/s)\,x) = (r/s)\,\psi\, x \qquad ((r/s)\,x \in C). \tag{3}$$

Moreover,

$$(r/s)\,\psi\, x \in Z \qquad ((r/s)\,x \in C), \tag{4}$$

because $\psi x \in Z$ and Z is a serving subgroup of the abelian torsion-free group D_k. From (3) and (4) we deduce that ψ induces an isomorphic mapping of C into Z. Hence, taking (1) and (2) into account, we conclude that there exists an isomorphic mapping of R_i into R_k. But this is impossible, because $i < k$. This proves that for $i < k$, $k \in M$, there exists no non-trivial homomorphism of D_i into D_k. Therefore, we deduce from (9.4) that (D) is partially ordered with respect to the set of all endomorphisms of G.

(12.6) Theorem. *Suppose that an abelian torsion-free group G has a canonical decomposition*

$$G = \sum_{a \in M} D_a, \tag{D}$$

and that at least one of the following two conditions is satisfied:

1. *G is a countable group;*
2. *the minimal condition holds in the partially ordered set $L(\mathrm{D}, \Omega)$, where Ω is the set of all idempotent endomorphisms of G.*

In order that any two direct decompositions of G have isomorphic refinements it is necessary and sufficient that for every $a \in M$ any two direct decompositions of D_a have isomorphic refinements.

Proof. By (12.5) (D) is partially ordered with respect to the set of all endomorphisms of G, since, by hypothesis, it is a canonical decomposition. Therefore, (D) is partially ordered with respect to the set of all idempotent endomorphisms of G. Theorem (12.6) now follows from Theorem (10.3).

(12.7) *If an abelian torsion-free group decomposes into the direct sum of isomorphic subgroups of rank 1, then any two direct decompositions of the group have isomorphic refinements.*

Proof. Let G be an abelian group that is decomposable into the direct sum of subgroups isomorphic to one and the same torsion-free group T of rank 1. By (11.2) every direct summand of G is a direct sum of subgroups isomorphic to T. Therefore, any two decompositions of G can be refined to decompositions into direct sums of subgroups isomorphic to T. But the number of direct summands in any decomposition of an abelian torsion-free group into the direct sum of subgroups of rank 1 is equal to the rank of the group. Therefore any two decompositions of G into the direct sum of subgroups isomorphic to T contain the same number of direct summands. Therefore, any two direct decompositions of G can be refined to decompositions into direct sums of subgroups isomorphic to T, and the refinements so obtained are isomorphic.

(12.8) *Any two direct decompositions of a completely decomposable abelian torsion-free group have isomorphic refinements.*

Proof. Let G be a countable completely decomposable abelian torsion-free group and

$$G = \sum_{a \in M} D_a \tag{D}$$

the canonical decomposition of G corresponding to an arbitrary decomposition of G into the direct sum of subgroups of rank 1. (Note that every torsion-free group of rank 1 is indecomposable). By (12.5) the canonical decomposition (D) is partially ordered with respect to the set of all endomorphisms of G, in particular, of all idempotent endomorphisms of G. Every direct summand D_a of (D) is a direct sum of isomorphic torsion-free subgroups of rank 1, so that, by (12.7) any two direct decompositions of D_a, $a \in M$, have isomorphic refinements. Therefore, we deduce from Theorem (12.6) that any two direct decompositions of G have isomorphic refinements.

(12.9) Theorem. *Any two direct decompositions of a completely decomposable countable group have isomorphic refinements.*

This theorem follows immediately from Theorems (12.2) and (12.8).

(12.10) Theorem. *Every direct summand of a completely decomposable countable group is also completely decomposable.*

(12.11) Theorem. *Every direct decomposition of a completely decomposable countable group can be refined to a decomposition into the direct sum of indecomposable subgroups of rank 1, and any two such decompositions are isomorphic.*

Theorem (12.10) and (12.11) follow immediately from Theorem (12.9).

(12.12) *Suppose that a completely decomposable abelian torsion-free group G satisfies the following condition:*

(a) *there exists at least one canonical decomposition*

$$G = \sum_{a \in M} D_a \tag{D}$$

corresponding to an arbitrary decomposition of G into the direct sum of subgroups of rank 1 for which the minimal condition holds in the partially ordered set $L(\mathrm{D}, \Omega)$, where Ω is the set of all idempotent endomorphisms of G. Then any two direct decompositions of G have isomorphic refinements.

Proof. As the canonical decomposition (D) corresponds to a decomposition of G into the direct sum of subgroups of rank 1, every direct summand D_a of (D) is a direct sum of isomorphic subgroups of rank 1. Therefore, by (12.7), any two direct decompositions of D_a, $a \in M$, have isomorphic refinements. Moreover, the canonical decomposition (D) satisfies condition 1 of Theorem (12.6), because it satisfies (a). Therefore, we deduce from Theorem (12.6) that any two direct decompositions of G have isomorphic refinements.

(12.13) Theorem. *Let H be a completely decomposable mixed abelian group and G the factor-group of H with respect to its maximal periodic subgroup. If G is either countable or satisfies condition* (a) *of proposition* (12.12), *then any two direct decompositions of G have isomorphic refinements.*

This theorem follows immediately from Theorems (12.2), (12.3), (12.8) and (12.12).

(12.14) Theorem. *Let H be a completely decomposable mixed abelian group and G the factor-group of H with respect to its maximal periodic subgroup. If G is either countable or satisfies condition* (a) *of proposition* (12.12), *then every direct decomposition of G can be refined to a decomposition into the direct sum of* *indecomposable subgroups of rank 1, and any two such decompositions are isomorphic.*

This theorem follows immediately from the preceding.

(12.15) **Theorem.** *Let H be a direct summand of a completely decomposable abelian group. If the factor-group of H with respect to its maximal periodic subgroup has not more than countable cardinal, then H is also completely decomposable.*

Proof. Let H be a direct summand of a completely decomposable abelian group A and

$$A = \sum_{i \in M} C_i \tag{C}$$

an arbitrary decomposition of A into the direct sum of subgroups of rank 1. We assume that the factor-group of H with respect to its maximal periodic subgroup has not more than countable cardinal. By $\{C_i\}_{i \in N}$ we denote the set of all those direct summands of (C) in which either H has a non-null component or every element has finite order. We put

$$B = \sum_{i \in N} C_i.$$

Obviously, B is a completely decomposable abelian group containing H and the maximal periodic subgroup of A. The factor-group of B with respect to its maximal periodic subgroup has not more than countable cardinal; this follows from the corresponding fact for H. Moreover, H is a direct summand of B because H is a subgroup of B which is a direct summand of A. Hence, it follows by (12.4) that H decomposes into the direct sum of subgroups of rank 1. This completes the proof.

A special case of Theorem (12.15) is

(12.16) **Theorem.** *Every countable direct summand of a completely decomposable abelian group is also completely decomposable.*

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NILPOTENT PRODUCTS OF GROUPS

O. N. GOLOVIN

The direct product and the free product are two operations on sets of groups which have in common a number of properties. Thus, both operations are associative and commutative. Each of the groups which these operations associate with, say, a given pair of groups as factors contains subgroups which are isomorphic to the factors, and these factors generate the appropriate product. In each case the intersection of one of these subgroups with the normal divisor generated by the other subgroup consists of the unit element only.

In this connection A. G. Kuroš in his book *Theory of Groups* proposed ([1], p. 350) the following group-theoretical problem: to define and classify all operations, if any, on a set of groups which share with the free and direct product the properties described above and yet are different from these two operations.

The importance of this problem derives above all from the fundamental part played by the operations of the free and direct product in the theory of groups. On the one hand these operations enable us to reduce the study of certain classes of groups to the study of classes of simpler or of better known groups (a classical example of this instance is the theory of Abelian groups with a finite number of generators); on the other hand (and this is indeed the most important aspect of the matter) these operations serve as a tool for the *construction* of new groups, or of new classes of groups, out of given groups.

Thus the problem is reduced, essentially, to the investigation of the possibility of defining new constructions similar to those of the direct and free product which would be a supplementary means in the study of various classes of groups and an instrument for the construction of new classes of groups. Of inherent interest also is the problem of whether it is possible to define "good" algebraic operations other than the direct product and the free product.

This paper is devoted to the construction of a denumerable number of algebraic operations (called nilpotent* operations) on sets of groups possessed of the properties mentioned above. Thus, a partial solution of the problem of A. G. Kuroš is obtained. Nilpotent operations of order zero are identical with direct products. Nilpotent operations of order one were studied by Levi [8], who refers to them as S-products. To judge by the review of Levi's paper in *Mathematical Reviews*,** Levi limited himself to the investigation of a product of two groups and did not pose the question of the associativity of S-products.

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* In the brief communication of the author (see [9]) on the essential results of this paper and also in his dissertation, *Central Products of Groups*, these operations are referred to as *central operations*.

** The author has not been able to acquaint himself with the original paper.

The first chapter of this paper is of auxiliary nature. It is a systematized collection of facts needed in the exposition following the first chapter and bearing almost exclusively on the theory of commutator subgroups of sets of subgroups of a group (see Ch. I, 1.3). These results are taken in part from Hall's paper [5] and from Zassenhaus' book [3]. Other than in the simplest cases referring to facts used in many papers, precise references are given in all appropriate places. The concept of a commutator subgroup involving, as it does in Hall's definition [5] (subsequently used by many authors), two groups is extended to an arbitrary set of subgroups. Accordingly, basic results are established for this general case. In §5 we study the properties of minimal central series introduced by Baer [6] and derive consequences of a certain theorem of Baer [6] which permit us to transfer many of the properties of direct products almost automatically to all nilpotent products (Ch. II, §6).

The construction and description of nilpotent products of groups do not solve Kuroš' problem completely. There remains the problem of finding *all* algebraic operations satisfying the required conditions. It is not correct to assume that the free product and the nilpotent products exhaust the set of all such operations. Recently, E. S. Lyapin communicated to the author a new example of such an operation.

CHAPTER I

Commutator subgroups of sets of subgroups

§1. Elementary definitions and notations.

1.1. The following notations will be used throughout this paper:

$G \cong H$—the group G is isomorphic to the group H;

$G \sim H$—the group G is homomorphic to the group H;

$\overline{A}^G$ (or simply $\overline{A}$)—the normal divisor generated by the subgroup A of the group G ("the normalized subgroup A");

$G = \{A_\alpha,\ \alpha \in \mathfrak{M}\}$ or $G = \{A, B, C, \cdots\}$—the group G is generated by its subgroups A_α where the index α ranges over a given set $\mathfrak{M}$, or, again, G is generated by its subgroups $A, B, C, \cdots$ (if specifically stipulated the symbols $A_\alpha, A, B, C, \cdots$ may denote arbitrary subsets in the group G);

$G = \Pi_{\alpha \in \mathfrak{M}} A_\alpha$—the group G is the product of its normal divisors A_α;

$G = \Pi^{\times}_{\alpha \in \mathfrak{M}} A_\alpha$ or $G = A \times B \times C \cdots$—the group G is the direct product of the groups A_α or, again, of the groups $A, B, C, \cdots$;

$G = \Pi^{*}_{\alpha \in \mathfrak{M}} A_\alpha$ or $G = A * B * C * \cdots$—the group G is the free product of the groups A_α or, again, of the groups $A, B, C, \cdots$.

1.2. If x and y are elements in a group G, then by the symbol (x, y) we shall mean the *commutator* of these two elements, i.e.,

$$(x, y) = x^{-1}y^{-1}xy.$$

The elements x, y shall be referred to as *components* of the commutator (x, y).

1.3. By the *commutator subgroup* $(A_\alpha, \alpha \in \mathfrak{M})$, or, simply, (A_α), *of a set of subgroups (subsets)* A_α of the group G (or: by the commutator subgroup $(A, B, C, \cdots)$ of a set of subgroups (subsets) $A, B, C, \cdots$ of the group G) we mean the subgroup of the group G generated by all commutators (a_α, a_β), $a_\alpha \in A_\alpha$, $a_\beta \in A_\beta$, $\alpha, \beta \in \mathfrak{M}$, $\alpha \neq \beta$ (by all commutators $(a, b), (a, c), (b, c), \cdots, a \in A, b \in B, c \in C, \cdots$ (cf. Hall[*] [5], p. 45). In particular, (G, G) stands for the commutator subgroup of G in the generally accepted sense. The commutator subgroup (G, A) shall be called the *principal commutator subgroup* of the subgroup A of the group G.

1.4. In order to stress that we are considering the commutator subgroup of a set of subsets of a definite group, we shall put the symbol denoting that group as an index after the symbol denoting the commutator subgroup. Thus, for instance, by $(A, B)_G$ is meant the commutator subgroup of the subgroups A and B of the group G. In particular we shall speak of the *free commutator subgroup* $(A_\alpha)_F$ *of the groups* A_α meaning the commutator subgroup of these groups in their free product. If the groups A_α are imbedded in a group H (i.e., these groups have isomorphic replicas in H), then we shall also speak of the commutator subgroup $(A_\alpha)_H$ of the groups A_α in the group H (or, simply, of their H-commutator subgroup).

1.5. Commutators of the form $((x, y), z)$ are called ([5], p. 43) *twofold commutators*; those of the form $((\cdots((x_0, x_1), x_2), \cdots), x_n)$ are called *n-fold commutators*. The elements $x_0, x_1, \cdots, x_n$ are called *components*; in particular, x_0, x_1 are called *inner components*.

1.6. Similarly, commutator subgroups of the form $((A, B), C)$ are called *twofold commutator subgroups*; those of the form $((\cdots((A_0, A_1), A_2), \cdots), A_n)$ are called *n-fold commutator subgroups* and the subgroups which appear in the expressions for commutator subgroups are called their *components*.

§2. Elementary properties of commutators and commutator subgroups.

2.1. The following are simple properties of the concepts introduced above:

2.1.1. $(x, y) = (y, x)^{-1}$.

2.1.2. $(xy, z) = y^{-1}(x, z)y\,(y, z) = (x, z)((x, z), y)(y, z)$.

2.1.3. $(x, yz) = (x, z)z^{-1}(x, y)z = (x, z)(z, (y, x))(x, y)$.

2.1.4. $(x, y^{-1}) = y(y, x)y^{-1}$; $(x^{-1}, y) = x(y, x)x^{-1}$.

2.1.5. $((x, y), z) =$

$$= (x, y)^{-1}(x, z)^{-1}(x,.(y, z))(y, z)^{-1}(x, y)(x, z)(y(x, z))^{-1}(y, z).$$

---

[*] Incidentally, Hall considers *simple* commutator subgroups of pairs of subgroups only and uses the symbol $(A, B, C, \cdots)$ to denote many-fold *commutator* subgroups.

Of these five relations the second and the third were, it seems, first proved by Hall ([5], p. 43) and the last one can be found in the book by Zassenhaus ([3], p. 60). All five relations can be checked by direct computation. As an example we prove the relation 2.1.5:

$$\begin{aligned}((x, y), z) &= (x, y)^{-1}z^{-1}(x, y)z = (x, y)^{-1}z^{-1}x^{-1}z \cdot z^{-1}y^{-1}z \cdot [z^{-1}y^{-1}z \cdot z^{-1}x^{-1}z]^{-1} = \\ &= (x, y)^{-1}(z, x)x^{-1}(z, y)y^{-1} \cdot [(z, v)y^{-1}(z, x)x^{-1}]^{-1} = \\ &= (x, y)^{-1}(x, z)^{-1}(x, (y, z)) \cdot (y, z)^{-1}x^{-1}y^{-1} \cdot [(z, y)(y, (x, z))(x, z)^{-1}y^{-1}x^{-1}]^{-1} = \\ &= (x, y)^{-1}(x, z)^{-1}(x (y, z)) \cdot (y, z)^{-1}(x, y)(x, z)(y, (x, z))^{-1}(y, z).\end{aligned}$$

2.2. In the definition 1.3 of the commutator subgroup $(A_\alpha, \alpha \in \mathfrak{M})$ we did not assume that $\mathfrak{M}$ could be ordered. However, if $\mathfrak{M}$ is an ordered set, it follows from the relation 2.1.1 that the commutator subgroup $(A_\alpha, \alpha \in \mathfrak{M})$ can be defined as the subgroup generated by the commutators (a_α, a_β), $\alpha < \beta$. In particular, the commutator subgroup (A, B) is generated by the commutators (a, b).

2.3. The operation of commutation of a pair of subgroups, i.e., the operation which associates with a pair of subgroups A, B of a group G their commutator subgroup (A, B), will play a major role in this paper. It follows from the definition of a commutator subgroup that this operation is commutative, $(A, B) = (B, A)$. However, it is easy to see that it is not associative. Thus for instance, if the group G is simple, non-abelian and the group A is an arbitrary, abelian subgroup of G, then $((G, A), A) = (G, A) = G$ and $(G, (A, A)) = 1$. (Here it is simplest to use the fact (see 3.2 below) that for an arbitrary subgroup A of a group G the principal commutator subgroup (G, A) of A is a normal divisor.)

2.4. We note a few almost obvious properties of commutator subgroups:

2.4.1. The subgroup A of the group G is a normal divisor if and only if $(G, A) \subseteq A$.

2.4.2. If $A_\alpha \subseteq B_\alpha$, then $(A_\alpha, \alpha \in \mathfrak{M}) \subseteq (B_\alpha, \alpha \in \mathfrak{M})$.

2.4.3. $(A, B) \subseteq \overline{A} \cap \overline{B}$. In particular, if N and M are normal divisors in G then $(N, M) \subseteq N \cap M$.

2.4.4. If N_α are normal divisors in G, then $\overline{(N_\alpha)} = (N_\alpha)$.

2.4.5. $\overline{(A_\alpha)} \subseteq (\overline{A}_\alpha)$. It is easy to see that we must not put an equality sign in this relation. If, for instance, we take, as in 2.3, for G a non-abelian, simple group and for A an arbitrary, abelian subgroup in G, then $\overline{(A, A)} = 1$ and $(\overline{A}, \overline{A}) = G$.

2.4.6. If N_α are normal divisors in G, then $(N_\alpha) = \Pi_{\alpha \neq \beta}(N_\alpha, N_\beta)$.

2.5. We noted in 2.3 that the operation of commutation of a pair of groups is non-associative. It is therefore necessary to pay attention to the disposition of brackets when dealing with complex commutator subgroups. (Complex commutator subgroups, i.e., commutator subgroups obtained by carrying out a number of operations of commutation, have been investigated by Hall ([5] pp. 42–48). Manyfold commutator subgroups (see 1.6) are a special case of complex commutator subgroups. Note that the relations 2.4.2–2.4.5 hold for arbitrary complex commutator subgroups. Thus, for instance, if $N_0, N_1, \cdots, N_n$ are normal divisors of a group G, then the n-fold commutator subgroup $((\cdots((N_0, N_1), N_2), \cdots), N_n)$ is a normal divisor and is contained in the intersection $N_0 \cap N_1 \cap \cdots \cap N_n$.

2.6. If ϕ is a homomorphic mapping of a group G onto a group G^* and A_α is a subgroup of G, then $(A_\alpha)\phi = (A_\alpha \phi)$.

Indeed, the image of a commutator is a commutator, and among the preimages of a commutator there always exists a commutator.

Assertion 2.6 holds for arbitrary, complex commutators.

2.7. If ϕ is a homomorphism of G onto G^* and A is a subgroup of G, then $(\overline{A}^G)\phi = \overline{A\phi}^{G^*}$.

This follows from the fact that a homomorphism carries a class of conjugate elements into a class of conjugate elements.

§3. Commutator subgroups of pairs of subgroups.

3.1. Lemma. *If a group G is generated by two of its subgroups A and B, then*

$$(A, B) = \overline{(A, B)} = (\overline{A}, \overline{B}).$$

Proof. 1) In order to show that $(A, B) = \overline{(A, B)}$ it obviously suffices to show that (A, B) contains, together with every commutator (a, b), all its conjugates in G. Indeed, if x is in A we have, by 2.1.2:

$$x^{-1}(a, b)\,x = (ax, b)\,(x, b)^{-1} \in (A, B),$$

and similarly, for y in B we have, by 2.1.3:

$$y^{-1}(a, b)\,y = (a, y)^{-1}(a, by) \in (A, B).$$

Since $G = \{A, B\}$, the inclusion: $g^{-1}(a, b)\,g \in (A, B)$, where g is an arbitrary element in G, can be proved by induction (on the length of the representation of the element g by means of factors from A and B).

2) To prove the equality $\overline{(A, B)} = (\overline{A}, \overline{B})$ we shall consider the factor group $G^* = G/\overline{(A, B)}$. This group is generated by the images A^* and B^* of the subgroups A and B under the natural homomorphism $G \sim G^*$; also, every element of A^* commutes with every element of B^* (see 2.6). Consequently, $\overline{A^*}^{G^*} = A^*$ and $\overline{B^*}^{G^*} = B^*$; since (see 2.7) $(\overline{A})^* = \overline{A}^*$ and $(\overline{B})^* = \overline{B}^*$, we have: $(\overline{A}, \overline{B})^* = ((\overline{A})^*, (\overline{B})^*) = (A^*, B^*) = 1$, i.e., $(\overline{A}, \overline{B}) \subseteq \overline{(A, B)}$. Since the reverse inclusion $(\overline{A}, \overline{B}) \supseteq \overline{(A, B)}$ is obvious, we conclude that $(\overline{A}, \overline{B}) = \overline{(A, B)}$.

3.2. Corollary. *The principal commutator subgroup of every subgroup A of a group G is a normal divisor and it coincides with the principal commutator subgroup* *of the normal divisor* $\overline{A}$:

$$(G, A) = \overline{(G, A)} = (G, \overline{A}).$$

3.3. Corollary. *If* A *is a subgroup of a group* G, *then*

$$\overline{A} = A\,(G, A).$$

Indeed, since $\overline{(G, A)} = (G, A)$ and $(G, A) \subseteq \overline{A}$ (see 2.4.1), it follows that $A(G, A)$ is a subgroup contained in $\overline{A}$. But this subgroup is a normal divisor in G for: $g^{-1} \cdot au \cdot g = a(a, g)u' = au''$, where $a \in A$, $g \in G$, $u, u', u'' \in (G, A)$.

3.4. Lemma. *If* $G = \{A, B\}$ *and* N *is a normal divisor in the subgroup* A, *then* $N(A, B)$ *is a normal divisor in* G.

Indeed, $N(A, B)$ is a subgroup (for (A, B) is a normal divisor). That $N(A, B)$ is a normal divisor follows from the fact that A and B generate G.

3.5. Corollary. *If* $G = \{A, B\}$ *then* $(G, A) = (A, A)(A, B)$ *and consequently*

$$\overline{A} = A\,(A, B).$$

Indeed, $(G, A) \supseteq (A, A)(A, B)$ so that, in view of Lemma 3.4, $(A, A)(A, B)$ is a normal divisor in G. Inasmuch as (a, a') and (a, b) are in $(A, A)(A, B)$, it follows by induction (on the length of the representation of an element g by means of factors from A and B) that (g, a) is also in $(A, A)(A, B)$ for an arbitrary g in G; for if $g = a'g'$, then $(g, a) = g'^{-1}(a', a)\, g'(g', a)$, and if $g = bg'$, then $(g, a) = g'^{-1}(b, a)\, g'(g', a)$ (see 2.1.2).

3.6. Lemma.* *If* N *is a normal divisor of the group* G *and* A *is any subgroup of the group* G *then*

$$(N, A) = (N, S_A),$$

where S_A *is an arbitrary system of generators of* A.

Proof. By definition (N, S_A) contains all commutators (n, s), where n is in N and s is an element of A whose representation by means of generators taken from S_A has length one. Let it be true that (N, S_A) contains all commutators (n, a'), where a' is an arbitrary element in A which admits of a shorter representation by means of generators in S_A than the given representation of the element a. Then the element a can be written as $a = a's$, $s \in S_A$ and, by 2.1.3,

$$(n, a) = (n, a's) = (n, s)\,(s, (a', n))\,(n, a') \in (N, S_A).$$

§4. Commutator subgroups of arbitrary sets of subgroups.

4.1. Lemma. *If the group* G *is generated by its subgroups* A_α, *then*

$$\overline{(A_\alpha)} = (\overline{A}_\alpha). \tag{4.1}$$

---

* This lemma is a special case of a theorem of Hall ([5], p. 45, Theorem 2.24).

The proof is a repetition of the second part of the proof of Lemma 3.1.

4.2. Remark. In contrast to the case when the group G is generated by two subgroups (see 3.1), the equality (4.1) cannot, in the general case, be supplemented by the equality $(A_\alpha) = (\overline{A_\alpha})$. This is shown by the following example.

Let $F = A * B * C$ be the free product of three cyclic groups of order two: $A = \{a\}$, $B = \{b\}$, $C = \{c\}$. *The commutator subgroup* (A, B, C) *is not a normal divisor of the group* F.

Indeed, $(A, B, C) = \{(A, B), (A, C), (B, C)\}$, i.e., every element in (A, B, C) can be represented in the form of a product of powers of commutators (a, b), (a, c) and (b, c); the k-th power of any of these commutators admits of no deletions (with respect to the elements a, b, c); as for the product of the commutator (a, b), say, with any of the commutators (a, c), (c, a), (b, c), or (c, b), deletions in that product are restricted to adjoining "exterior" elements. It follows that the first three letters of an arbitrary element in (A, B, C) (of course, in its irreducible representation in terms of a, b, c) must coincide with the first three letters of one of the six commutators (a, b), (b, a), (a, c), (c, a), (b, c), (c, b). Consequently, the element $c^{-1}(a, b)c = cababc$ is not in (A, B, C).

4.3. Lemma.* *If we are given a group* G, *a subgroup* A *in* G, *a normal divisor* N *in* G *and a set of normal divisors* M_α *in* G, *then*

$$\Big(A \cdot \prod_\alpha M_\alpha, N\Big) = (A, N) \cdot \prod_\alpha (M_\alpha, N).$$

Proof. Inasmuch as $M_\alpha \subset \prod_\alpha M_\alpha$, it follows that $(A, N) \cdot \prod_\alpha (M_\alpha, N) \subseteq (A \cdot \Pi_\alpha M_\alpha, N)$. We shall prove the reverse inclusion. Every element g in $A \cdot \Pi_\alpha M_\alpha$ can be represented in the form of a finite product $g = m_{\alpha_1} \cdots m_{\alpha_s} a$, where $m_{\alpha_i} \in M_{\alpha_i} (i = 1, \cdots, s)$, $a \in A$. But for an arbitrary $n \in N$ we have, by 2.1.2, $(g, n) = (m_{\alpha_1} \cdots m_{\alpha_s} a, n) = a^{-1}(m_{\alpha_1} \cdots m_{\alpha_s}, n)a \cdot (a, n)$, where

$$a^{-1}(m_{\alpha_1} \ldots m_{\alpha_s}, n)a \in (M_{\alpha_1} \ldots M_{\alpha_s}, N)$$

(since the M_α are normal divisors). Assuming, by induction, that for all elements g' which admit of a shorter representation than g, (g', n) is in $(A, N)\prod_\alpha (M_\alpha, N)$, we conclude that $(g, n) \in (A, N) \prod_\alpha (M_\alpha, N)$.

4.4. Remark. The above results show that the two abgebraic operations—multiplication and commutation—defined on a set of normal divisors of a fixed group G (the two operations associate with every set of normal divisors in G two definite normal divisors in G) possess the following properties:

---

* For a special case of this lemma where the set of normal divisors consists of a single normal divisor, see Zassenhaus ([3], p. 58).

I. Multiplication: a) commutative, b) associative;

II. Commutation: a) commutative;

III. Commutation is connected with multiplication by the distributive law.

4.4.1. In 2.3 we gave an example of the non-associativity of the operation of commutation when applied to arbitrary subgroups. It is not difficult to construct an example which shows this operation to be non-associative even if we restrict ourselves to normal divisors. Thus, if $\mathfrak{A}_4$ denotes the alternating group of degree four, and K denotes its commutator subgroup (the so-called four-group of Klein), then

$$((\mathfrak{A}_4, \mathfrak{A}_4), K) = (K, K) = 1, \quad (\mathfrak{A}_4, (\mathfrak{A}_4, K)) = (\mathfrak{A}_4,.K) = K,$$

for, since K is of index *3*, it is clearly not the center. (It is a well-known fact that a factor group by the center cannot be cyclic.)

4.4.2. The distributive property of commutation with respect to multiplication of normal divisors of a fixed group G (formulated in 4.3) does not hold if instead of normal divisors we consider all subgroups and instead of multiplication—the generation of a subgroup by a given set of subgroups. In other words, the assertion (analogue of Lemma 4.3):"given: a subgroup A and a certain set of subgroups A_α of a group G, it follows that $(\{A_\alpha\}, A) = \{(A_\alpha, A)\}$" does not hold.

Indeed, let us return to the example of the free product $F = A * B * C$ of three cyclic subgroups of order two. We showed in 4.2 that the commutator subgroup $(A, B, C)_F$ does not contain elements whose irreducible form begins with abc, say. All the more so, these elements are not contained in the subgroup $\{(A, C), (B, C)\}$. On the other hand, the commutator subgroup $(A * B, C)$ contains the commutator $(ba, c) = abcbac$.

4.5. Lemma. *Let G be a group. Let there be given in G a set of normal divisors M_α, $\alpha \in \mathfrak{M}$. Let $\mathfrak{M}$ be subdivided in an arbitrary manner into non-intersecting subsets: $\mathfrak{M} = \bigcup_{\omega\in\Omega} \mathfrak{M}_\omega$. Then*

$$(M_\alpha, \alpha\in\mathfrak{M}) = \Big(\prod_{\alpha_\omega\in\mathfrak{M}_\omega} M_{\alpha_\omega}, \omega\in\Omega\Big)\cdot \prod_{\omega\in\Omega} (M_{\alpha_\omega}, \alpha_\omega \in\mathfrak{M}_\omega),$$

where the multiplication $\Pi_{\omega\in\Omega}$ extends over all ω for which $\mathfrak{M}_\omega$ contains more than one element.

Indeed, by 2.4.6, $(M_\alpha, \alpha\in\mathfrak{M}) = \Pi_{\alpha\neq\beta}(M_\alpha, M_\beta)$. Using 2.4.6 and Lemma 4.3 to transform the first factor in the product on the right:

$$\Big(\prod_{\alpha_\omega \in \mathfrak{M}_\omega} M_{\alpha_\omega}, \omega \in \Omega\Big) = \prod_{\omega \neq \omega'}\Big(\prod_{\alpha_\omega \in \mathfrak{M}_\omega} M_{\alpha_\omega}, \prod_{\alpha_{\omega'} \in \mathfrak{M}_{\omega'}} M_{\alpha_{\omega'}}\Big) =$$

$$\prod_{\omega \neq \omega'} \prod_{\alpha_\omega \in \mathfrak{M}_\omega} \prod_{\alpha_{\omega'} \in \mathfrak{M}_{\omega'}} (M_{\alpha_\omega}, M_{\alpha_{\omega'}}) ,$$

and expanding each of the factors in the product $\Pi_{\omega \in \Omega}(M_{\alpha_\omega}, \alpha_\omega \in \mathfrak{M}_\omega)$ in accordance with 2.4.6, we find that the right side of the equation which appears in the statement of our lemma is also equal to $\Pi_{\alpha \neq \beta}(M_\alpha, M_\beta)$.

4.6. Corollary. *In the special case of three arbitrary subgroups A, B, C in G we have*

$$(\bar{A}, \bar{B}, \bar{C}) = (\bar{A}, \bar{B})(\bar{A}\bar{B}, \bar{C}).$$

4.7. Lemma.* *If $A_1, A_2, \cdots, A_n$ are normal divisors in a group G and A_0 is an arbitrary subgroup with a system of generators S_0, then*

$$((\ldots((A_0, A_1), A_2), \ldots), A_n) = \{((\ldots((s_0, a_1), a_2), \ldots), a_n)\},$$

i.e., the commutator subgroup on the left is generated by commutators of the form indicated on the right, where s_0 ranges over S_0 and each a_i ranges over the subgroup A_i $(i = 1, 2, \cdots, n)$.

This lemma is an immediate consequence of Lemma 3.6, and is proved by induction on n.

4.8. Lemma.** *If A, B, C are normal divisors of a group G, then*

$$((A, B), C) \subseteq ((B, C), A)((C, A), B).$$

Proof. By Lemma 4.7, $((A, B), C) = \{((a, b), c)\}$, and by the relation 2.1.5,

$$((a, b), c) = (a, b)^{-1}(a, c)^{-1}(a, (b, c)) \cdot (b, c)^{-1}(a, b)(a, c)(b, (a, c))^{-1}(b, c),$$

or, employing the equivalence $\bmod\ [((B, C), A) \cdot ((C, A), B)]$,

$$((a, b), c) \equiv (a, b)^{-1}(a, c)^{-1}(b, c)^{-1}(a, b)(a, c)(b, c) \equiv$$

$$\equiv (a, b)^{-1}(b, c)^{-1}(a, b)(b, c) \equiv (a, b)^{-1}(a, b) = 1,$$

which proves our lemma.

§5. On minimal central series.

5.1. Baer ([6], p. 348) defines *the minimal central series of a group G determined by a subgroup A* to be the decreasing series of normal divisors whose terms are given by the recursive formulas:

---

* Cf. [5], p. 45, Theorem 2.24.

** Cf. [3], p. 60 or [5], p. 47.

$${}_0A = \overline{A}, \quad {}_lA = (G, {}_{l-1}A) \text{ for } l > 0.$$

For the sake of brevity we shall sometimes refer to this series as the *G-A series.*

A well-known example of a minimal central series is the G-G-series, called *the lower central series of the group G:*

$${}^0G \supseteq {}^1G \supseteq \ldots \supseteq {}^lG \supseteq \ldots,$$

where ${}^0G = G$ and ${}^lG = (G, {}^{l-1}G)$ for $l > 0$.

5.2. Theorem 3.2 implies that ${}_1A = (G, A)$ which, in turn, enables us to give the definition of a G-A-series in a somewhat different form, namely: ${}_0A = \overline{A}$, ${}_1A = (G, A)$, ${}_lA = (G, {}_{l-1}A)$ for $l > 1$.

5.3. **Lemma.** *If there is given in a group G an arbitrary set of subgroups* A_α, *then*

$${}_l\{A_\alpha\} = \prod_\alpha {}_lA_\alpha \quad (l = 0, 1, \ldots).$$

The theorem holds trivially for $l = 0$. Let us assume that the theorem holds for $l - 1$. Then, by Lemma 4.3,

$${}_l\{A_\alpha\} = (G, {}_{l-1}\{A_\alpha\}) = (G, \prod_\alpha {}_{l-1}A_\alpha) = \prod_\alpha {}_lA_\alpha.$$

5.4. **Lemma.** *If A is an arbitrary subgroup in G, then*

$$({}^{l-1}G, A) \subseteq {}_lA \quad (l = 1, 2, \ldots).$$

Indeed, if $l = 1$, then $({}^0G, A) = (G, A) = {}_1A$. Let the lemma be true for $l - 1$ and for an arbitrary subgroup in G. Using Lemma 4.8 we find that

$$({}^{l-1}G, A) \subseteq ((G, {}^{l-2}G), \overline{A}) \subseteq (({}^{l-2}G, \overline{A}), G) \cdot ((\overline{A}, G), {}^{l-2}G) \subseteq$$

$$({}_{l-1}A, G) \cdot ({}_1A, {}^{l-2}G) \subseteq {}_lA \cdot {}_lA = {}_lA.$$

5.5. **Lemma.** *If* $G = \{A_\alpha\}$, *then*

$${}_l(A_\alpha) = {}_l(\overline{A}_\alpha) \quad (l = 0, 1, \ldots).$$

Indeed, by definition, ${}_k(A_\alpha) = {}_k[\overline{(A_\alpha)}]$ which in turn equals ${}_k(\overline{A}_\alpha)$ (by Lemma 4.1).

5.6. **Lemma.** *If* $G = \{A_\alpha, \alpha \in \mathfrak{M}\}$ *and the set* $\mathfrak{M}$ *of indices is divided into non-intersecting subsets:* $\mathfrak{M} = \bigcup_{\omega \in \Omega} \mathfrak{M}_\omega$ *and if* $G_\omega = \{A_{\alpha_\omega}, \alpha_\omega \in \mathfrak{M}_\omega\}$, *then*

$${}_k(A_\alpha, \alpha \in \mathfrak{M}) = {}_k(G_\omega, \omega \in \Omega) \cdot \prod_{\omega \in \Omega} {}_k(\overline{A}_{\alpha_\omega}, \alpha_\omega \in \mathfrak{M}_\omega).$$

Indeed, Lemmas 5.5, 4.5, and 5.3 imply

$$
{}_k(A_\alpha,\ \alpha \in \mathfrak{M}) = {}_k(\bar{A}_\alpha,\ \alpha \in \mathfrak{M}) =
$$
$$
{}_k(\bar{G}_\omega,\ \omega \in \Omega) \cdot \prod_{\omega \in \Omega} {}_k(\bar{A}_{\alpha_\omega},\ \alpha_\omega \in \mathfrak{M}_\omega) = {}_k(G_\omega,\ \omega \in \Omega) \cdot \prod_{\omega \in \Omega} {}_k(\bar{A}_{\alpha_\omega},\ \alpha_\omega \in \mathfrak{M}_\omega).
$$

5.7. Lemma. *Let G be a group. Let N be a normal divisor in G. Let there be given an arbitrary set of subgroups A_α, $\alpha \in \mathfrak{M}$, in G. Then*

$$
(A_\alpha N,\ \alpha \in \mathfrak{M}) \subseteq (A_\alpha,\ \alpha \in \mathfrak{M}) \cdot {}_1N.
$$

Proof. The commutator subgroup on the left is generated by the commutators $(a_\alpha \cdot u,\ a_\beta \cdot u')$, where $a_\alpha \in A_\alpha$, $a_\beta \in A_\beta$ $(\alpha, \beta \in \mathfrak{M};\ \alpha \neq \beta)$, $u, u' \in N$. Using the relations 2.1.2–2.1.3 and bearing in mind that N is a normal divisor, we get:

$$
(a_\alpha \cdot u,\ a_\beta \cdot u') = (a_\alpha,\ a_\beta \cdot u')\,((a_\alpha, a_\beta \cdot u'), u)\,(u, a_\beta \cdot u') = (a_\alpha, a_\beta \cdot u') \cdot {}_1u =
$$
$$
(a_\alpha, u')\,(u', (a_\beta, a_\alpha))\,(a_\alpha, a_\beta) \cdot {}_1u = (a_\alpha, a_\beta) \cdot {}_1u', \text{ where } {}_1u, {}_1u' \in {}_1N.
$$

This proves our lemma.

5.8. Lemma. *If N is a normal divisor in the free factor A of the free product $F = A * B$, then*

$$
F / [\bar{N}^F \cdot {}_k(A, B)_F\,] \cong F' / {}_k(A/N, B)_{F'},
$$

where $F' = [A/N] * B$.

Proof. If M is an arbitrary normal divisor in F, then, using the second theorem on isomorphisms ([2], p. 137), and the Theorem of Dyck ([1], p. 86), we get:

$$
F / \bar{N}M \cong [F/\bar{N}] / [\bar{N}M/\bar{N}] \cong [[A/N] * B] / M',
$$

where M' denotes the homomorphic image of the normal divisor M under the natural homomorphism $A * B \sim [A/N] * B$. The assertion of our lemma now follows from the remark 2.6 with M replaced by the commutator subgroup ${}_k(A, B)_F$.

5.9. As is well-known (see, e.g., [1], p. 105), by the *upper* or *increasing central series of a group G* is meant the series of normal divisors

$$
Z_0 = 1 \subseteq Z_1 \subseteq \ldots \subseteq Z_n \subseteq \ldots,
$$

where Z_n/Z_{n-1} is the center of the factor group G/Z_{n-1}.

5.10. Lemma. *The upper central series of a group G can be defined as the series whose n-th term is the (unique for every n) maximal subgroup which determines in the group a minimal central series whose n-th term equals the unit element.*

Indeed, the definition of an upper central series implies that its n-th term Z_n is a (unique) maximal subgroup satisfying the condition $(G, Z_n) \subseteq Z_{n-1}$. Consequently, ${}_nZ_n = 1$. If H is an arbitrary subgroup of G satisfying the condition ${}_nH = 1$, then ${}_{n-1}H \subseteq Z_1$, ${}_{n-2}H \subseteq Z_2$, $\ldots$, ${}_1H \subseteq Z_{n-1}$ and, finally, $H \subseteq Z_n$, which proves the lemma.

5.11. Corollary. *In a group without center none of the minimal central series (except for the trivial one defined by the unit element) reaches the unit element.*

5.12. The following theorem proved by Baer ([6], p. 357, Theorem 3) plays an important part in the study of minimal central series:

Let N *be a normal divisor of finite index in a group* G. *Then the group* $[(G, G) \cap N]/(N, G)$ *is finite and the orders of its elements are divisors of the number* n.

We do not give a proof of this theorem. We shall use it to derive a corollary which is of importance in subsequent considerations (Ch. II, §6).

5.13. Corollary. *If* N *is a normal divisor of finite index in* G *and if* N *is contained in the commutator subgroup* (G, G), *then all factor groups* $G/_iN$ *are finite and in their orders there appear only such prime numbers as divide the number* n.

Proof. By assumption the theorem holds for the factor group $G/_0N$. Let it hold for $G/_{i-1}N$. Using Baer's theorem we conclude that the factor group $_{i-1}N/_iN$ is finite and that the prime divisors of its order divide n. Consequently, the theorem holds also for the group $G/_iN$.

5.14. By a *nilpotent** *group* we shall mean a group G whose lower central series consists of a finite number of terms. If $^lG \neq 1$ and $^{l+1}G = 1$, we shall say that the *nilpotent group* G *belongs to the* l*-th class.*** Clearly, the nilpotent groups of class zero are the abelian groups; the nilpotent groups of class 1 are often referred to as *metabelian groups* (cf., e.g., Šmidt, [4], p. 133).

5.15. Consequence of Theorem 5.12. *A nilpotent group* G *which has a system* S *of generators of finite order is locally finite. Also, the prime divisors of the orders of all elements of* G *are restricted to the prime divisors of the orders of the elements of* S. *In particular, if the orders of all elements of* S *are powers of a single prime number* p *then* G *is a* p*-group.*

Proof. We select in G an arbitrary, finite system of elements $g_1, g_2, \cdots, g_n$. The subgroup generated by these elements is contained in the group $G' = \{s_1, s_2, \cdots, s_m\}$, where $s_1, s_2, \cdots, s_m$ is the system of all those generators taken from S which appear in an arbitrary but fixed representation of the elements $g_1, g_2, \cdots, g_n$. Since the factor group $G'/(G', G')$ is finite (as an abelian group with a finite number of generators of finite order), the factor groups $G'/^iG'$ must also be finite for an arbitrary natural number i (this follows from corollary 5.13). But since G' is a subgroup of a nilpotent group it must itself be nilpotent; therefore, for some l, $G'/^lG' = G'$. As for the assertion relating to the prime factors of the orders of the elements of G, it is also a consequence of 5.13, for it holds in the factor group

---

* In the classification employed by Kuroš and Černikov ([7], p. 49) this is a Z_0-group.

** Such a group is usually said to belong to the $(l + 1)$-st class.

$G'/(G', G')$.

5.16. Corollary.* *A periodic nilpotent group is locally finite.*

CHAPTER II

On a class of associative operations on a set of groups

§1. Regular products of groups.

1.1. Definition. If $G = \{A_\alpha, \alpha \in \mathfrak{M}\}$ and $A_\alpha \cap \overline{B}_\alpha = 1$ for all $\alpha \in \mathfrak{M}$, where $B_\alpha = \{A_\beta, \beta \in \mathfrak{M} \setminus \alpha\}$, then we shall say that G is *a regular product* $\Pi^{\Delta}_{\alpha \in \mathfrak{M}} A_\alpha$ *of the subgroups* A_α or that G *is regularly generated by the subgroups* A_α, $\alpha \in \mathfrak{M}$, which are referred to as *regular factors* of the group G.

The direct product and the free product of an arbitrary set of groups are examples of regular products.

1.2. Theorem. *If a group* G *is generated by a set of subgroups* A_α, $\alpha \in \mathfrak{M}$, *and the set* $\mathfrak{M}$ *of indices is ordered, then* $G = \Pi^{\Delta}_{\alpha \in \mathfrak{M}} A_\alpha$ *if and only if every element* g *in* G *has a "regular" representation*

$$g = a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_n} u, \tag{1.2}$$

where $a_{\alpha_i} \in A_{\alpha_i}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, $u \in (\overline{A}_\alpha, \alpha \in \mathfrak{M})$, *which is unique to within components* a_{α_i} *equal to* 1.

Proof. Let $G = \Pi^{\Delta}_{\alpha \in \mathfrak{M}} A_\alpha$. Then the *existence* of a representation of the form (1.2) is a consequence of the fact that $G = \{A_\alpha, \alpha \in \mathfrak{M}\}$. We must still prove the *uniqueness* of such a representation. Let us assume that $g = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n} u = a'_{\alpha_1} a'_{\alpha_2} \cdots a'_{\alpha_n} u'$ (clearly, it may be required beforehand that the indices of the components coincide in pairs). Writing this equality in the form $a^{-1}_{\alpha_1} a'_{\alpha_1} = (a_{\alpha_2} \cdots a_{\alpha_n} u)(a'_{\alpha_2} \cdots a'_{\alpha_n} u')^{-1}$, we conclude on the basis of definition 1.1 that $a^{-1}_{\alpha_1} a'_{\alpha_1} = 1$, i.e., $a_{\alpha_1} = a'_{\alpha_1}$ and $a_{\alpha_2} \cdots a_{\alpha_n} u = a'_{\alpha_2} \cdots a'_{\alpha_n} u'$. Similarly, $a_{\alpha_2} = a'_{\alpha_2}$, and so on.

Conversely, let us assume that every element g in G can be uniquely represented in the form (1.2). Fix an arbitrary α_0 and put $B_{\alpha_0} = \{A_\beta, \beta \in \mathfrak{M} \setminus \alpha_0\}$. Since $G = \{A_{\alpha_0}, B_{\alpha_0}\}$, it follows by 3.5, Ch.I, that $\overline{B}_{\alpha_0} = B_{\alpha_0} \cdot (A_{\alpha_0}, B_{\alpha_0}) = B_{\alpha_0} \cdot (\overline{A}_{\alpha_0}, \overline{B}_{\alpha_0}) = B_{\alpha_0} \cdot (\overline{A}_{\alpha_0}, \overline{B}_{\alpha_0})(\overline{A}_\beta, \beta \in \mathfrak{M} \setminus \alpha_0) = B_{\alpha_0} \cdot (\overline{A}_\alpha, \alpha \in \mathfrak{M})$ (see 4.5, Ch.I). Hence, if $g \in A_{\alpha_0} \cap \overline{B}_{\alpha_0}$, i.e., $g = a_{\alpha_0} = a_{\alpha_1} \cdots a_{\alpha_n} u$, where $a_{\alpha_0} \in A_{\alpha_0}$, $a_{\alpha_i} \in A_{\alpha_i}$, $\alpha_i \in \mathfrak{M} \setminus \alpha_0$, $i > 0$, $u \in (\overline{A}_\alpha, \alpha \in \mathfrak{M})$, then, in view of the uniqueness of the representation (1.2) for each element, $g = 1$.

1.3. If $G = \Pi^{\Delta}_{\alpha \in \mathfrak{M}} A_\alpha$ and the element g has the regular representation (1.2), then we call the components $a_{\alpha_1}, \cdots, a_{\alpha_n}$ its *regular components* and u, its *supplementary component*. To find the regular component of g in the factor A_{α_0} we can, obviously, proceed as follows: take an arbitrary representation of the element g by means of the a_α and replace in it all a_α with $\alpha \neq \alpha_0$ by unity; the element thus obtained is the α_0-th component of g. We infer from this that a change in the ordering of the set $\mathfrak{M}$ does not imply a change in the regular components of the elements.

---

* This theorem can be found, among others, in the paper by Kuroš and Černikov ([7], pp. 42 and 55).

When two elements are multiplied their corresponding regular components are multiplied; in particular, in the case of mutually inverse elements, corresponding regular components are also mutually inverse. From this it follows that the orders of the regular components of each element are divisors of the order of that element.

1.4. For an arbitrary subgroup H of a regular product $G = \Pi^{\Delta} A_\alpha$ the collection of those regular components of the elements of H which have the same subscript forms a subgroup in the corresponding regular factor. We shall call this subgroup *a component of* H in the corresponding regular factor. It follows from 1.3 that if H is a normal divisor in G, then each of its components is a normal divisor in the corresponding factor.

1.5. Corollary. *The subgroup generated by subgroups of the regular factors of a regular product* $G = \Pi^{\Delta} A_\alpha$ *is itself a regular product of those subgroups.*

Indeed, the regular representation of the form (1.2) is uniquely defined for each of the elements of this subgroup.

1.6. Corollary. *If* $G = \Pi^{\Delta}_{\alpha\in\mathfrak{M}} A_\alpha$, *then*

$$G/(\overline{A}_\alpha,\ \alpha\in\mathfrak{M}) \cong \prod_{\alpha\in\mathfrak{M}}^{\times} A_\alpha.$$

Indeed, this isomorphism is determined by the correspondence

$$g = a_{\alpha_1}a_{\alpha_2}\dots a_{\alpha_n}u \to a_{\alpha_1}a_{\alpha_2}\dots a_{\alpha_n}.$$

1.7. In case of a finite $\mathfrak{M}$ we shall sometimes write the product $\Pi_{i=1}^{n\,\Delta} A_i$ in the form $A_1 \Delta A_2 \Delta \cdots \Delta A_n$. We consider in somewhat greater detail the case of two regular factors: $G = A \Delta B$. Definition 1.1 is now formulated as follows: a group G is said to be *a regular product of two of its subgroups* A and B if $G = \{A, B\}$ and $A \cap \overline{B} = \overline{A} \cap B = 1$. In the terminology of the theory of extensions of groups ([3], p. 59) this means that G is a regular product of two of its subgroups A and B if and only if G is generated by A and B and, furthermore, G is an extension of $\overline{A}$ by means of B and, at the same time, of $\overline{B}$ by means of A (which means that every coset of G with respect to $\overline{A}$ (or $\overline{B}$) contains exactly one element of B (of A)).

Since $G = \{A, B\}$ implies (see Lemma 3.1, Ch. I) that $(A, B) = (\overline{A, B})$, it follows that in the case of two factors Theorem 1.2 takes the following simplified form:

is called a *defining kernel* of the regular product $\Pi^{\Delta} A_{\alpha}$.

1.14. The following theorem affords some insight into a portion of the totality of regular products of two groups.

Theorem. *The free commutator subgroup* $(A, B)_F$ *of two arbitrary groups of finite orders* k *and* l *is a free group of rank* $(k-1)(l-1)$; *in addition, the set of all "free" commutators* (a, b), *where* a *ranges over all elements of* A *different from* 1, *and* b *ranges over all elements of* B *different from* 1, *forms a set of free generators of that free group.*

Proof. All free commutators $(a, b) = a^{-1}b^{-1}ab$ are obviously different from 1. By a *word* with respect to the free commutators we shall mean any finite product of free commutators comtaining no two inverse terms (a, b) and $(a, b)^{-1} = (b, a)$ in a row. We shall show that no such non-empty word equals the unit element. Let us call two commutators (a, b) and (a', b') (as well as the commutators (b, a) and (b', a')) "of the same type", and two commutators (a, b) and (b', a') "of different types". It can be easily checked that two adjoining commutators of the same type admit of no cancellations. As for two adjoining commutators of different types, (a, b) and (b', a'), say, cancellations cannot reach their centers; indeed, even if $b = b'$, (by the definition of a word) $a \neq a'$. This insures the "freeness" of the free commutators as generators of the group $(A, B)_F$.

1.15. **Remark.** If the groups A and B are infinite, then their free commutator subgroup $(A, B)_F$ is a group whose rank equals the product of the cardinalities of the groups A and B, respectively.

Indeed, in this case the "free" commutators (a, b) form a free basis for the group $(A, B)_F$.

1.16. **Remark.** We recall that in the case of two groups A and B the free commutator subgroup $(A, B)_F$ is already a normal divisor. It follows, in particular, that the free product $A * B$ of two groups of (finite) order k and l, respectively, is an extension of the free group of rank $(k-1)(l-1)$ by means of the direct product of those groups. (If A and B are infinite and their cardinalities are, respectively, κ and λ, then $A * B$ is an extension of the free group of rank $\kappa\lambda$ by means of the direct product $A \times B$.)

1.17. In the sequel the general properties of regular products play a conspicuous but nevertheless auxiliary role. The two most interesting questions bearing on regular products to which the author does not know the answer are:

1) Is regular multiplication associative in the sense that if $G = \Pi_{\alpha}^{\Delta} G_{\alpha}$ and $G_{\alpha} = \Pi_{\beta}^{\Delta} A_{\alpha,\beta}$, then $G = \Pi_{\alpha,\beta}^{\Delta} A_{\alpha,\beta}$?

2) Theorem 1.14 can be extended to the case of an arbitrary number of groups A_{α}; in that case the free commutator subgroup $(A_{\alpha})_F$ will be a free group. But in

1.8. Theorem. *A group G is a regular product of two of its subgroups A and B if and only if each element in G has a unique regular representation of the form*

$$g = a\,b\,u, \tag{1.8}$$

where $a \in A$, $b \in B$, $u \in (A, B)$.

1.9. Theorem. *If $G = A \,\Delta\, B$ and one of the factors, A, say, is a normal divisor, then B is a normal divisor and G is the direct product of A and B.*

Indeed, since $\bar{A} = A$, it follows that (see 2.4.3, Ch. I)

$$(A, B) \subseteq \bar{A} \cap \bar{B} = A \cap \bar{B} = 1,$$

i.e., every element A commutes with every element of B so that A and B are normal divisors and G is their direct product.

1.10. Definition. If the kernel of a homomorphism of a regular product $\Pi^{\Delta}_{\alpha \in \mathfrak{M}} A_\alpha$ is contained in the commutator subgroup $(\bar{A}_\alpha,\ \alpha \in \mathfrak{M})$, then this homomorphism is called *regular* and the corresponding factor group is referred to as a *regular factor group*.

1.11. Lemma. *A regular homomorphic image G^* of a group $G = \Pi^{\Delta}_{\alpha \in \mathfrak{M}} A_\alpha$ is itself a regular product of its subgroups A^*_α, which are isomorphic images of the subgroups A_α under the considered regular homomorphism.*

Proof. It is clear that $G^* = \{A^*_\alpha,\ \alpha \in \mathfrak{M}\}$. Since the kernel N of the homomorphism is in $(\bar{A}_\alpha,\ \alpha \in \mathfrak{M})$ it follows that $A_\alpha \cap N = 1$, i.e., $A^*_\alpha \cong A_\alpha$. Finally, $A_\alpha \cap \bar{B}_\alpha = 1$, where $B_\alpha = \{A_\beta,\ \beta \in \mathfrak{M} \setminus \alpha\}$, implies (see 2.7, Ch. I) that $A^*_\alpha \cap \bar{B}^*_\alpha = 1$, since $N \subseteq (\bar{A}_\alpha) \subset \bar{B}_\alpha$.

1.12. Theorem. *Every regular product $G = \Pi^{\Delta}_{\alpha \in \mathfrak{M}} A_\alpha$ is a regular factor group of the free product $F = \Pi^{*}_{\alpha \in \mathfrak{M}} A_\alpha$ of those very groups. Conversely, every regular factor group of the free product $F = \Pi^{*}_{\alpha \in \mathfrak{M}} A_\alpha$ is a regular product of those very groups.*

Proof. That G is a factor group of the free product F follows from Dyck's theorem (see, for instance, [1], p. 86). There remains to show that the kernel N of the homomorphism is contained in the normalized (see 1.1, Ch. I) free commutator subgroup $(\bar{A}_\alpha)_F$. Let the element $f = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n} u_f$, where $a_{\alpha_i} \in A_{\alpha_i}$, $u_f \in (\bar{A}_\alpha)_F$, be in N. The image of the element f under the natural homomorphism of F onto G is the element $g = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n} u_g$, $u_g \in (\bar{A}_\alpha)_G$ with "the same" regular components a_{α_i}. But since f is in N, it follows that $g = 1$, whence $a_{\alpha_1} = a_{\alpha_2} = \cdots = a_{\alpha_n} = 1$, i.e., $N \subseteq (\bar{A}_\alpha)_F$.

The converse follows immediately on the basis of Lemma 1.11 in view of the regularity of the free product.

1.1.3. Definition. The kernel of a regular homomorphism of $\Pi^{*} A_\alpha$ onto $\Pi^{\Delta} A_\alpha$

contrast to the case of two groups where we have $(A, B) = \overline{(A, B)}$, in this more general case $(A_\alpha) \neq \overline{(A_\alpha)}$ (see 4.2, Ch. I). The question arises as to whether the normalized free commutator subgroup $\overline{(A_\alpha)}_F$ is a free group.

§2. Fully regular operations on a set of groups.

2.1. Definition. A *fully regular* Δ-*product of groups* is said to be given if there is associated with an arbitrary set of groups A_α, $\alpha \in \mathfrak{M}$ (note that this set may contain an arbitrary number of isomorphic groups) a certain group denoted by $\Pi^{\Delta}_{\alpha\in\mathfrak{M}} A_\alpha$ and furthermore, the following conditions hold:

a) the group $\Pi^{\Delta}_{\alpha\in\mathfrak{M}} A_\alpha$ is a regular product (see 1.1) of its subgroups which are isomorphic to the corresponding groups A_α, $\alpha \in \mathfrak{M}$ (*regularity of* Δ-*products*);

b) if $G = \prod^{\Delta}_{\alpha\in\mathfrak{M}} A_\alpha$ and $A_\alpha = \prod^{\Delta}_{\beta_\alpha\in\mathfrak{M}_\alpha} A_{\alpha\beta_\alpha}$, $\alpha\in\mathfrak{M}$, then $G = \prod^{\Delta}_{\substack{\beta_\alpha\in\mathfrak{M}_\alpha \\ \alpha\in\mathfrak{M}}} A_{\alpha\beta_\alpha}$ (*associativity of* Δ-*products*).

Since $\mathfrak{M}$ is not assumed to be ordered we may speak of the commutativity of Δ-multiplication. In view of the regularity property we may think of each A_α as being contained in $\Pi^{\Delta}_{\alpha\in\mathfrak{M}} A_\alpha$.

2.2. The conditions in definition 2.1 are compatible.

Indeed, it is easy to see that the direct product and the free product are fully regular.

2.3. We note that if we restrict ourselves to finite fully regular products, then it is obvious that it suffices to study products *of pairs* of groups. In the finite case definition 2.1 runs as follows:

Definition. We shall say that a *finite fully regular* Δ-*product of groups* has been given if there is associated with every pair of (not necessarily distinct) groups A_α and A_β a certain group, denoted by $A_\alpha \Delta A_\beta$, and the following conditions hold:

a) the group $A_\alpha \Delta A_\beta$ is a regular product of two of its subgroups which are isomorphic to A_α and A_β, respectively (regularity of Δ-products);

b) $(A_\alpha \Delta A_\beta) \Delta A_\gamma \cong A_\alpha \Delta (A_\beta \Delta A_\gamma)$ (associativity of Δ-products);

c) $A_\alpha \Delta A_\beta \cong A_\beta \Delta A_\alpha$ (commutativity of Δ-products).

2.4. On the basis of Theorems 1.6 and 1.12 we may say that fully regular operations occupy an intermediate position between the direct product of groups and the free product of groups. Also, every such operation defines a system of defining kernels (1.13) for every set of groups A_α. In what follows, we construct a series of fully regular products by means of a suitable choice of a system of defining kernels which insures the associativity of the multiplication involved.

§3. Definition of nilpotent products of groups.

3.1. Definition. The terms of the minimal central series (see 5.1, Ch. I) of the free product $F = \Pi^*_{\alpha \in \mathfrak{M}} A_\alpha$, defined by the free commutator subgroup $\overline{(A_\alpha)_F}$ are called *free commutator subgroups* of the set of groups A_α, $\alpha \in \mathfrak{M}$; namely, the k-th term ${}_k(A_\alpha)_F$ is called the *k-th free commutator subgroup* (of the groups A_α).

Thus, the 0-th free commutator subgroup of a set of groups A_α, $\alpha \in \mathfrak{M}$ is the normalized commutator subgroup $\overline{(A_\alpha)_F}$; in particular, in the case of two groups A and B the 0-th free commutator subgroup is the free commutator subgroup $(A, B)_F$.

3.2. Definition. By the *k-th nilpotent product* of a set of groups A_α, $\alpha \in \mathfrak{M}$ ($k = 0, 1, \cdots$) we mean the factor group of their free product by their k-th free commutator subgroup:

$$\prod_{\alpha \in \mathfrak{M}}^{(k)} A_\alpha = \left[\prod_{\alpha \in \mathfrak{M}}^{*} A_\alpha\right] \Big/ {}_k(A_\alpha)_F\,.$$

In case of a finite number of factors we shall also employ the notation

$$A_1\,(k)\,A_2\,(k)\ldots(k)\,A_n.$$

We shall call the commutator subgroup (A_α) of the groups A_α in $\Pi^{(k)} A_\alpha$ their *k-th nilpotent commutator subgroup* and we shall denote it by $(A_\alpha)^{(k)}$. Clearly, (see 2.6, Ch. I)

$$(A_\alpha)^{(k)} = (A_\alpha)_F \cdot {}_k(A_\alpha)_F / {}_k\,(A_\alpha)_F$$

3.3 We conclude on the basis of Theorem 1.12 that all nilpotent products are regular. In §5 we prove the associativity of nilpotent products.

We note that the *0-th nilpotent product is the direct product of groups.*

3.4. Definition. By the *(k, i)-th nilpotent commutator* subgroup of a set of groups A_α, $\alpha \in \mathfrak{M}$ we mean the i-th term of the minimal central series of the group $\Pi^{(k)}_{\alpha \in \mathfrak{M}} A_\alpha$ determined by the k-th nilpotent commutator subgroup $(A_\alpha)^{(k)}$, i.e., the subgroup ${}_i(A_\alpha)^{(k)}$.

3.5. In view of 2.6 and 2.7 in Chapter I we have, for an arbitrary $i \leq k$,

$${}_i(A_\alpha)^{(k)} = {}_i(A_\alpha)_F / {}_k(A_\alpha)_F\,;$$

therefore, the minimal central series under investigation terminates after at most k steps:

$${}_0(A_\alpha)^{(k)} = \overline{(A_\alpha)}^{(k)} \supseteq {}_1(A_\alpha)^{(k)} \supseteq \ldots \supseteq {}_k(A_\alpha)^{(k)} = 1.$$

3.6. Theorem. *The factor group of the k-th nilpotent product of a set of groups A_α, $\alpha \in \mathfrak{M}$ by the (k, l)-th ($0 \leq l \leq k$) nilpotent commutator subgroup ${}_l(A_\alpha)^{(k)}$ is isomorphic to the l-th nilpotent product of this very same set of groups:*

$$\prod^{(k)} A_\alpha / {}_l(A_\alpha)^{(k)} \cong \prod^{(l)} A_\alpha \quad (0 \leqslant l \leqslant k).$$

In view of 3.5 this theorem is a direct consequence of the second theorem on isomorphisms (see, for instance, [2], p. 185).

§4. Some lemmas on nilpotent products.

4.1. Lemma. *If* $G = A(k)B$ *and* $A = \{A_\alpha,\ \alpha \in \mathfrak{M}\}$, *then*

$${}_l(\overline{A}^G_\alpha,\ \alpha\in\mathfrak{M})_G = {}_l(\overline{A}^A_\alpha,\ \alpha\in\mathfrak{M})_G \quad \textit{for} \quad l \geqslant k-1.$$

Proof. Noting that $\overline{A}^G_\alpha \subseteq \overline{A}^A_\alpha(A,\ B)$ and using Lemma 5.7, Ch. I., we obtain:

$$(\overline{A}^A_\alpha)_G \subseteq (\overline{A}^G_\alpha)_G \subseteq (\overline{A}^A_\alpha(A,B))_G \subseteq (\overline{A}^A_\alpha)_G \cdot {}_1(A,\ B)_G.$$

If we now use Lemma 5.3, Ch. I, we find that

$${}_l(\overline{A}^A_\alpha)_G \subseteq {}_l(\overline{A}^G_\alpha)_G \subseteq {}_l(\overline{A}^A_\alpha)_G \cdot {}_{l+1}(A\,B)_G.$$

The assertion of our lemma now follows from the fact that ${}_k(A,\ B)^{(k)} = 1$ (see 3.5).

4.2. Corollary. *If* $G = A(k)B$ *and* $A = \{A_1,\ A_2\}$, *then*

$${}_l(\overline{A}^G_1,\ \overline{A}^G_2)_G = {}_l(A_1,\ A_2)_G \quad \textit{for} \quad l \geqslant k-1.$$

Indeed, in this case (see 3.1, Ch. I) $(\overline{A}^A_1,\ \overline{A}^A_2) = (A_1,\ A_2)$.

4.3. Lemma 5.4, Ch. I, implies that if we are given in a group G an arbitrary set of subgroups A_α, then

$$({}^{l-1}G,\ {}_l(A_\alpha)) \subseteq {}_{l+i}(A_\alpha) \ \text{ for } i > 0 \text{ and } l \geq 0.$$

In particular, we obtain the following lemma:

4.4. Lemma. *If* $G = \Pi^{(k)}_\alpha A_\alpha$, *then*

$$({}^{k-1}G,\ (A_\alpha)) = 1.$$

Indeed, ${}_k(A_\alpha)^{(k)} = 1$.

4.5. Lemma. *If* $G = A(k)B$ *and* ${}^0A \supseteq {}^1A \supseteq \cdots \supseteq {}^nA \supseteq \cdots$ *is the lower central series of the group* A, *i.e.,* ${}^0A = A$, ${}^nA = (A,\ {}^{n-1}A)$ *for* $n > 0$, *then*

$$({}^nA,\ B) \subseteq {}_n(A,\ B).$$

Proof. If $n = 0$ the assertion is trivial since $({}^0A,\ B) = (A,\ B)$ and ${}_0(A,\ B) = (A,\ B)$. Let the theorem hold for all integers less than n. Using 2.1.5, Ch. I, we obtain for the elements $a \in A$, ${}^{n-1}a \in {}^{n-1}A$, $b \in B$:

$$((a,{}^{n-1}a),\ b) = (a,{}^{n-1}a)^{-1}(a,\ b)^{-1}(a,\ ({}^{n-1}a,\ b))\,({}^{n-1}a,\ b)^{-1}\cdot$$
$$\cdot(a,\ {}^{n-1}a)(a,\ b)\,({}^{n-1}a,\ (a,\ b))^{-1}\,({}^{n-1}a,\ b).$$

By the induction assumption, $({}^{n-1}a, b) \in {}_{n-1}(A, B)$. Consequently, $(g, ({}^{n-1}a, b)) \in {}_n(A, B)$, i.e., the element $({}^{n-1}a, b)$ commutes mod ${}_n(A, B)$ with an arbitrary element of the group G. Also, by 4.3, $({}^{n-1}a, (a, b)) \in {}_n(A, B)$ and

$$((a, {}^{n-1}a), (a, b)) \in {}_{n+1}(A, B) \subseteq {}_n(A, B).$$

Consequently, $(a, {}^{n-1}a)$ also commutes mod ${}_n(A, B)$ with (a, b). Hence, the right side of the expression for the element $((a, {}^{n-1}a), b)$ turns out to be equal to 1 mod ${}_n(A, B)$, i.e., $((a, {}^{n-1}a), b) \in {}_n(A, B)$. If in addition, we note that (in view of the relation 2.12 in Ch. I) for arbitrary ${}^na_1, {}^na_2 \in {}^nA$, $b \in B$

$$({}^na_1 \cdot {}^na_2, b) = ({}^na_1, b)(({}^na_1, b), {}^na_2)({}^na_2, b),$$

and that the elements $(a, {}^{n-1}a)$ generate ${}^nA$, then the proof of this lemma is easily completed using induction.

4.6. Corollary. *If* $G = A(k)B$ *and* ${}^0A \supseteq {}^1A \supseteq \cdots \supseteq {}^nA \supseteq \cdots$ *is the lower central series of the group* A, *then*

$$(B, {}^kA) = 1,$$

i.e., every element of each of the factors A *and* B *commutes with every element of the* k*-th term of the lower central series of the other factor.*

Indeed, it suffices to recall that ${}_k(A, B)^{(k)} = 1$.

4.7. Lemma. *If* $G = A(k)B$ *and* $a, a' \in A$ *and* $g_i = a_i b_i u_i$ $(i = 1, 2, \cdots, l)$ *are* l *arbitrary elements in* G *given in their regular representations, then*

$$((\dots((a, a'), g_1), \dots), g_l) = ((\dots((a, a'), a_1), \dots), a_l) \cdot {}_lu,$$

where ${}_lu \in {}_l(A, B)$.

Proof. We shall denote elements in ${}_i(A, B)$ by ${}_iu$ or by ${}_iu'$. For the sake of convenience we rewrite equation 2.13, Ch. I, in the form:

$$(x, yz) = (x, z)(x, y)((x, y), z).$$

(This equality is best verified directly.) Using it, we find that

$$\begin{aligned} ((a, a'), g_1) &= ((a, a'), a_1b_1 \cdot u_1) = \\ &= ((a, a'), u_1)((a, a'), a_1b_1) \cdot (((a, a'), a_1b_1), u_1) = \\ &= {}_2u \cdot ((a, a'), a_1 \cdot b_1) \cdot {}_3u = {}_2u \cdot ((a, a'), b_1)((a, a'), a_1)(((a, a'), a_1), b_1) \cdot {}_3u = \\ &= {}_2u \cdot {}_1u \cdot ((a, a'), a_1) \cdot {}_2u' \cdot {}_3u = ((a, a'), a_1) \cdot {}_1u'. \end{aligned}$$

In proving that certain elements are contained in certain commutator subgroups ${}_i(A, B)$, we made use of Lemmas 4.3 and 4.5. Let us assume that the correctness of our lemma has already been proved for all $i < l$, in particular, that

$((\dots((a, a'), g_1), \dots), g_{l-1}) = {}^la \cdot {}_{l-1}u$, where ${}^la = ((\dots((a, a'), a_1), \dots), a_{l-1})$.

Then

$$((\ldots((a,\ a'),\ g_1),\ \ldots),\ g_l) = ({}^l a\cdot{}_{l-1}u,\ g_l) =$$
$$= ({}^l a\cdot{}_{l-1}u,\ u_l)\,({}^l a\cdot{}_{l-1}u, a_l b_l)\,(({}^l a\cdot{}_{l-1}u,\ a_l b_l),\ u_l).$$

Using 2.1.2, Ch. I, we find further that

$$({}^l a\cdot{}_{l-1}u,\ u_l) = ({}^l a,\ u_l)\,(({}^l a,\ u_l),\ {}_{l-1}u)\,({}_{l-1}u,\ u_l) = {}_{l+1}u'\cdot{}_{2l+2}u\cdot{}_{l+1}u'' = {}_{l+1}u,$$

$$({}^l a\cdot{}_{l-1}u,\ a_l b_l) = ({}^l a,\ a_l b_l)\,(({}^l a,\ a_l b_l),\ {}_{l-1}u)\,({}_{l-1}u,\ a_l b_l) =$$
$$= ({}^l a,\ a_l b_l)\cdot{}_{2l+1}u\cdot{}_l u' = ({}^l a,\ b_l)\,({}^l a,\ a_l)\,(({}^l a,\ a_l),\ b_l)\cdot{}_{2l+1}u\cdot{}_l u' =$$
$$= {}_l u''\cdot({}^l a,\ a_l)\cdot{}_{l+1}u'''\cdot{}_{2l+1}u\cdot{}_l u' = ({}^l a,\ a_l)\cdot{}_l u,$$
$$(({}^l a\cdot{}_{l-1}u,\ a_l b_l),\ u_l) = (({}^l a,\ a_l)\cdot{}_l u,\ u_l) =$$
$$= (({}^l a,\ a_l),\ u_l)\cdot((({}^l a,\ a_l),\ u_l),\ {}_l u)\cdot({}_l u,\ u_l) = {}_{l+2}u'\cdot{}_{2l+3}u\cdot{}_{l+2}u'' = {}_{l+2}u.$$

Thus, the element which is of interest to us is equal to the product

$${}_{l+1}u\cdot({}^l a,\ a_l)\cdot{}_l u\cdot{}_{l+2}u = ({}^l a,\ a_l)\cdot{}_l\bar{u},$$

which proves our lemma.

4.8. Corollary. *If* $G = \Pi_\alpha^{(k)}$ *and if there is given in* A_{α_0} *an arbitrary set of subgroups* C_γ, *then*

$${}_l(C_\gamma)_G \subseteq {}_l(C_\gamma)_{A_{\alpha_0}}\cdot{}_l(A_\alpha) \quad (l = 0, 1, \ldots).$$

Indeed, in view of Lemma 4.7, Ch. I, the subgroup ${}_l(C_\gamma)_G$ is generated by commutators of the form

$$((\ldots((c_\gamma,\ c_\delta),\ g_1),\ \ldots), g_l), \quad \text{where} \quad c_\gamma \in C_\gamma,\ c_\delta \in C_\delta, \gamma \neq \delta,$$

and $g_1, \cdots, g_l \in G$. By Lemma 4.7, Ch. II, all such commutators are contained in

$${}_l(C_\gamma)_{A_{\alpha_0}}\cdot{}_l(A_{\alpha_0},\ B_0), \quad \text{where} \quad B_0 = \prod_{\alpha \neq \alpha_0}^{(k)} A_\alpha.$$

Consequently, ${}_l(C_\gamma)_G \subseteq {}_l(C_\gamma)_{A_{\alpha_0}}\cdot{}_l(A_{\alpha_0},\ B_0)$. But in view of 4.3, Ch. I, and 4.1, Ch. I:

$${}_l(A_{\alpha_0},\ B_0) = {}_l(\bar{A}_{\alpha_0},\ \bar{B}_0) = {}_l(\bar{A}_{\alpha_0}, \prod_{\alpha \neq \alpha_0} \bar{A}_\alpha) = {}_l\{\prod_{\alpha \neq \alpha_0} (\bar{A}_{\alpha_0},\ \ \bar{A}_\alpha)\} \subseteq {}_l(\bar{A}_\alpha) = {}_l(A_\alpha).$$

4.9. Corollary. *If* $G = \Pi_\alpha^{(k)} A_\alpha$ *and if there is given in* A_{α_0} *an arbitrary set of subgroups* C_γ, *then* ${}_l(C_\gamma)_G = {}_l(C_\gamma)_{A_{\alpha_0}}$ *for* $l \geq k$.

Indeed, for $l \geq k$, ${}_l(A_\alpha)^{(k)} = 1$.

4.10. Lemma. *If there is given in the group* $G = A(k)B$ *a normal divisor* N *contained in* A, *then*

$$[A\,(k)\,B]/N \cong [A/N]\,(k)\,B.$$

Proof. By the second theorem on isomorphisms (see [2], p. 185) $[A(k)B]/N \cong [A * B]/\overline{N}^F \cdot {}_k(A, B)_F$ which is, in turn, isomorphic to (see Lemma 5.8, Ch. I) the factor group

$$[(A/N) * B]/{}_k(A/N, B)_{[(A/N) * B]} = (A/N)(k)B.$$

§5. Theorem concerning the associativity of nilpotent products.

5.1. Theorem. *If* $G = \Pi^{(k)}_{\alpha \in \mathfrak{M}} A_\alpha$ *and if the set of indices* $\mathfrak{M}$ *is written as the union of two non-intersecting subsets* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$, *then the subgroups* $G_1 = \{A_\beta, \beta \in \mathfrak{M}_1\}$ *and* $G_2 = \{A_\gamma, \gamma \in \mathfrak{M}_2\}$ *are the k-th nilpotent products* $G_1 = \Pi^{(k)}_{\beta \in \mathfrak{M}_1} A_\beta$ *and* $G_2 = \Pi^{(k)}_{\gamma \in \mathfrak{M}_2} A_\gamma$ *of the subgroups which generate them and the group* G *is the k-th nilpotent product of the subgroups* G_1 *and* G_2:

$$G = G_1(k)G_2.$$

Proof. We put $F = \Pi^*_{\alpha \in \mathfrak{M}} A_\alpha$, $F_1 = \Pi^*_{\beta \in \mathfrak{M}_1} A_\beta$, $F_2 = \Pi^*_{\gamma \in \mathfrak{M}_2} A_\gamma$. Now, using 5.6, Ch. I, the second theorem on isomorphisms, 2.6 and 2.7, Ch. I, 4.1 and 4.9, Ch. II, 5.5, Ch. I, and (twice) 4.10, Ch. II, we obtain the following chain of equalities and isomorphisms:

$$\begin{aligned}
G &= F/{}_k(A_\alpha)_F = F/[{}_k(F_1, F_2)_F \cdot {}_k(\overline{A}^F_\beta, \beta \in \mathfrak{M}_1)_F \cdot {}_k(\overline{A}^F_\gamma, \gamma \in \mathfrak{M}_2)_F] \cong \\
&\cong [F/{}_k(F_1, F_2)]/[[{}_k(F_1, F_2)_F \cdot {}_k(\overline{A}^F_\beta)_F \cdot {}_k(\overline{A}^F_\gamma)_F]/{}_k(F_1, F_2)] = \\
&= [F_1(k)F_2]/[{}_k(\overline{A}^{F_1(k)F_2}_\beta)_{F_1(k)F_2} \cdot {}_k(\overline{A}^{F_1(k)F_2}_\gamma)_{F_1(k)F_2}] = \\
&= [F_1(k)F_2]/[{}_k(\overline{A}^{F_1}_\beta)_{F_1} \cdot {}_k(\overline{A}^{F_2}_\gamma)_{F_2}] = \\
&= [F_1(k)F_2]/[{}_k(A_\beta)_{F_1} \cdot {}_k(A_\gamma)_{F_2}] \cong \\
&\cong [F_1/{}_k(A_\beta)_{F_1}](k)[F_2/{}_k(A_\gamma)_{F_2}] = G'_1(k)G'_2.
\end{aligned}$$

It is not difficult to see that under the above isomorphism of G onto $G'_1(k)G'_2$ every group A_α, considered as a factor in the group G, is mapped upon itself, now considered as a factor in either G'_1 or G'_2, however. Thus, in particular, G_1 is mapped onto G'_1 and G_2 onto G'_2.

5.2. The converse theorem. *If* $G = G_1(k)G_2$ *and* $G_1 = \Pi^{(k)}_{\beta \in \mathfrak{M}_1} A_\beta$ *and* $G_2 = \Pi^{(k)}_{\gamma \in \mathfrak{M}_2} A_\gamma$, *then* $G = \Pi^{(k)}_{\alpha \in \mathfrak{M}} A_\alpha$, *where* $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$.

Indeed, consider the product $G' = \Pi^{(k)}_{\alpha \in \mathfrak{M}} A_\alpha$. By the preceding theorem, $G' = G'_1(k)G'_2$, where $G'_1 = \Pi^{(k)}_{\beta \in \mathfrak{M}_1} A_\beta$ and $G'_2 = \Pi^{(k)}_{\gamma \in \mathfrak{M}_2} A_\gamma$. Consequently $G \cong G'$, which proves the theorem.

5.3. Theorem. *If* $G = \Pi^{(k)}_{\alpha \in \mathfrak{M}} A_\alpha$ *and* $\mathfrak{M} = \{0, 1, 2, \cdots, a, \cdots\}$ *is well-ordered and of order-type* μ, *then* G *is the union of an increasing (transfinite) chain of subgroups:*

$$A_0 \subset A_0 (k) A_1 \subset \dots \subset \prod_{\alpha < \lambda}^{(k)} A_\alpha \subset \cdots,$$

where λ *takes on all values smaller than* μ*.*

Proof. By Theorem 5.1, G actually contains such a chain of subgroups. Since every element in G has a representation consisting of a finite number of elements taken from the subgroups A_a, it follows that evey element is contained in some term of the chain. This proves the assertion that G is the union of the groups of the chain under consideration.

5.4. The converse theorem. *If there is given a set of groups* A_a, $a \in \mathfrak{M}$, *and* $\mathfrak{M} = \{0, 1, \cdots, a, \cdots\}$ *is well-ordered and of order-type* μ, *then the union* G *of the increasing chain of groups* $G_\lambda = \Pi^{(k)}_{a<\lambda} A_a$ $(\lambda = 1, 2, \cdots, \nu, \cdots < \mu)$ *is itself the k-th nilpotent product of all the groups* A_a: $G = \Pi^{(k)}_{a<\mu} A_a$.

Indeed, let G' denote the k-th nilpotent product of the groups A_a. By the preceding theorem, G' is the union of an increasing chain of subgroups isomorphic to the groups G_λ and embedded one in the other in the manner of the groups G_λ. It follows that the groups G and G' are isomorphic.

5.5. Theorem on associativity. *If* $G = \Pi^{(k)}_{a \in \mathfrak{M}} A_a$ *and* $\mathfrak{M} = \bigcup_{\omega \in \Omega} \mathfrak{M}_\omega$ *and the set* Ω *is well-ordered, then*

$$G_\omega = \{A_{\alpha_\omega}, \alpha_\omega \in \mathfrak{M}_\omega\} = \prod_{\alpha_\omega \in \mathfrak{M}_\omega}^{(k)} A_{\alpha_\omega}, \ \omega \in \Omega, \quad \text{and} \quad G = \prod_{\omega \in \Omega}^{(k)} G_\omega.$$

Proof. We shall prove the theorem by means of transfinite induction on the order-type λ_Ω of the set Ω. Theorem 5.1 proves the correctness of our theorem for the case $\lambda_\Omega = 2$. We assume its validity for all $\lambda < \lambda_\Omega$.

Case 1: λ_Ω is not a limit ordinal. Then $\lambda_\Omega - 1 = \lambda'$ exists. We put $\bigcup_{\omega=1}^{\lambda'} \mathfrak{M}_\omega = \mathfrak{M}^*$; thus, $\mathfrak{M} = \mathfrak{M}^* \cup \mathfrak{M}_{\lambda_\Omega}$ and, by Theorem 5.1,

$$G = \Big(\prod_{\beta \in \mathfrak{M}^*}^{(k)} A_\beta \Big) (k) \Big(\prod_{\gamma \in \mathfrak{M}_{\lambda_\Omega}}^{(k)} A_\gamma \Big).$$

Noting that by the induction assumption

$$\prod_{\beta \in \mathfrak{M}^*}^{(k)} A_\beta = \prod_{\omega=1}^{\lambda'} {}^{(k)} \Big(\prod_{\alpha_\omega \in \mathfrak{M}_\omega}^{(k)} A_{\alpha_\omega} \Big),$$

we find that

$$G = \left(\prod_{\omega=1}^{\lambda'} {}^{(k)} \Big(\prod_{\alpha_\omega \in \mathfrak{M}_\omega}^{(k)} A_{\alpha_\omega} \Big) \right) (k) \left(\prod_{\gamma \in \mathfrak{M}_{\lambda_\Omega}}^{(k)} A_\gamma \right).$$

Making use of Theorem 5.2, we find that the theorem on associativity holds in the case under consideration.

Case 2: λ_Ω is a limit ordinal. We put $\mathfrak{N}_\lambda = \bigcup_{\omega<\lambda} \mathfrak{M}_\omega$ for all $\lambda < \lambda_\Omega$. We form in G the chain of subgroups

$$\prod_{\alpha\in\mathfrak{N}_1}^{(k)} A_\alpha \subset \prod_{\alpha\in\mathfrak{N}_2}^{(k)} A_\alpha \subset \ldots \subset \prod_{\alpha\in\mathfrak{N}_\lambda}^{(k)} A_\alpha \subset \ldots.$$

Clearly, the union of this chain of subgroups coincides with the whole group G. Also by the induction assumption,

$$\prod_{\alpha\in\mathfrak{N}_\lambda}^{(k)} A_\alpha = \prod_{\omega<\lambda}^{(k)} \Big(\prod_{\alpha_\omega\in\mathfrak{M}_\omega}^{(k)} A_{\alpha_\omega} \Big).$$

Hence, by Theorem 5.4, G, as the union of an increasing chain of subgroups $\Pi^{(k)}_{\omega<\lambda} (\Pi^{(k)}_{a_\omega\in\mathfrak{M}_\omega} A_{a_\omega})$ is itself equal to the product $\Pi^{(k)}_{\omega\in\Omega} (\Pi^{(k)}_{a_\omega\in\mathfrak{M}_\omega} A_{a_\omega})$. This completes the proof of our theorem.

5.6. The converse theorem. *If* $G = \Pi^{(k)}_{\omega\in\Omega} G_\omega$ *and the set* Ω *is well-ordered, and* $G_\omega = \Pi^{(k)}_{a_\omega\in\mathfrak{M}_\omega} A_{a_\omega}$, $\omega \in \Omega$, *then* $G = \Pi^{(k)}_{a\in\mathfrak{M}} A_a$, *where* $\mathfrak{M} = \bigcup_{\omega\in\Omega} \mathfrak{M}_\omega$.

The proof is analogous to the proof of Theorem 5.2.

5.7. Conclusion. *All nilpotent products of groups are fully regular operations.*

Indeed, their regularity follows from definition 3.2, and Theorem 5. 6 proves their associativity.

§6. Some properties of nilpotent products.

We shall now study general properties which all nilpotent products have in common. We shall be primarily interested in determining which properties of the factors are carried over to their nilpotent products.

Most of the proofs given below are based on Corollary 5.13, Ch. I, to Baer's theorem (see 5.12, Ch. I).

6.1 Theorem. *If* $G = \Pi^{(k)}_{a\in\mathfrak{M}} A_a$ *and if we single out in each of the factors* A_a *a certain subgroup* A'_a *then the subgroup* $G' = \{A'_a, a \in \mathfrak{M}\}$ *is isomorphic to a regular factor group of the k-th nilpotent product* $\Pi^{(k)}_{a\in\mathfrak{M}} A'_a$.

Proof. Consider the subgroup $F' = \Pi^*_{a\in\mathfrak{M}} A'_a$ *in the free product* $F = \Pi^*_{a\in\mathfrak{M}} A_a$. Since $G = F/{}_k(A_a)_F$, it follows that $G' = [F' \cdot {}_k(A_a)_F]/{}_k(A_a)_F \cong F'/[F' \cap {}_k(A_a)_F]$. But clearly, ${}_k(A'_a)_{F'} \subseteq {}_k(A_a)_F$. Hence, $F' \cap {}_k(A_a)_F \supseteq {}_k(A'_a)_{F'}$, i.e., $G' \cong \Pi^{(k)}_{a\in\mathfrak{M}} A'_a/V$, where $V = [F' \cap {}_k(A_a)_F]/{}_k(A'_a)_{F'}$. As for the assertion that $V \subseteq {}_0(A'_a)^{(k)}$, we need only apply Theorem 1.12 to prove it; for, in view of 1.5, G' is a regular product of the subgroups A_a, $a \in \mathfrak{M}$.

6.1.1. Remark. In the case of a direct product or of a free product we get a stronger result; namely, if $G = \Pi^* A_a$, then $G' = \Pi^* A'_a$ and if $G = \Pi^\times A_a$, then $G' = \Pi^\times A'_a$. In the general case, however, the conclusion of Theorem 6.1 cannot be strengthened in the sense of these two examples for it is possible to construct a 1-st nilpotent product $G = \Pi^{(1)} A_\alpha$, such that a certain subgroup $G' = \{A'_\alpha \subseteq A_\alpha, \alpha \in \mathfrak{M}\}$ in this product is the 0-th (i.e., direct) rather than the 1-st nilpotent product of the subgroups A'_α.

6.2. **Lemma.** *The normalized k-th nilpotent commutator subgroup* $\overline{(A_\alpha)^{(k)}}$ *of an arbitrary set of groups* A_α *is a nilpotent group of class lower than k.*

Indeed, we showed in 3.5 that the minimal central series of the k-th nilpotent product of the groups A_α, determined by the commutator subgroup $(A_\alpha)^{(k)}$ reaches the unit element in at most k steps. This is all the more true of the lower central series of the group $\overline{(A_\alpha)^{(k)}}$ for, clearly, $\overline{{}^{l}(A_\alpha)^{(k)}} \subseteq {}_{l}(A_\alpha)^{(k)}$.

6.3. By the *characteristic of a group* we mean the set of all prime numbers which appear in the orders of the elements of that group. If the group contains at least one element of infinite order, then we include the number 0 in the characteristic of the group. By the sum of the characteristics of a number of groups we mean their set-theoretical sum. Two characteristics are called *co-prime* if 1 is their only common number.

Theorem. *The k-th nilpotent product G of a finite number of finite groups* A_α *is a finite group whose characteristic is the sum of the characteristics of the factors.*

Indeed, since this theorem holds for the direct product of these groups, we may conclude on the basis of 5.13, Ch. I, that it also holds for an arbitrary nilpotent product of these groups.

6.4. **Theorem.** *The k-th nilpotent product* $G = \Pi_\alpha^{(k)} A_\alpha$ *of locally finite groups is locally finite and the characteristic of the group G equals the sum of the characteristics of the factors.*

Proof. Let $g_1, g_2, \cdots, g_n$ be an arbitrary, finite system of elements in G. The regular representations of these elements involve only (both as regular components and as supplementary components from the commutator subgroup $\overline{(A_\alpha)}$) a finite number of elements from the factors A_α. Let these elements generate the subgroups $B_1 \subseteq A_{\alpha_1}, B_2 \subseteq A_{\alpha_2}, \cdots, B_m \subseteq A_{\alpha_m}$. By Theorem 6.1, the subgroup $B = \{B_1, B_2, \cdots, B_m\}$ is isomorphic to a factor group of the k-th nilpotent product of the groups $B_1, B_2, \cdots, B_m$. This product is finite (see the preceding theorem) since the local finiteness of the factors A_α implies the finiteness of the subgroups $B_1, B_2, \cdots, B_m$.

The assertion pertaining to the characteristic of the group is also a consequence of Theorem 6.3.

6.5. **Lemma.** *If* $G = A(k)B$ *and if we select in the factors A and B two elements a and b of finite orders l and m, then their commutator* (a, b) *is of finite* *order n and every prime divisor of n divides either l or m.*

Indeed, the subgroup $G' = \{a, b\}$ is (see 6.1) a factor group of the k-th nilpotent product $\{a\}(k)\{b\}$. Applying Theorem 6.3 to that product, we obtain the required result.

6.6. Theorem. *The k-th nilpotent product $G = \Pi^{(k)}_\alpha A_\alpha$ of periodic groups is periodic. Also, the characteristic of the group G is equal to the sum of the characteristics of the factors.*

Proof. The periodicity of the direct product of periodic groups implies that some power of every element in G is contained in $\overline{(A_\alpha)}$. But it has been shown in 6.2 that $\overline{(A_\alpha)}$ is a nilpotent group. Since $\overline{(A_\alpha)}$ has (see 6.5) a system S of generators of finite orders, it follows by 5.15, Ch. I, that $\overline{(A_\alpha)}$ is locally finite.

The assertion concerning the characteristic also follows from 5.15, Ch. I.

6.7. Corollary. *The k-th nilpotent product of p-groups (for the same p) is a p-group.*

6.8. Theorem. *If $G = \Pi^{(k)}_\alpha A_\alpha$ then*

$${}^lG = \prod_\alpha {}^l A_\alpha \cdot {}_{l-1}(A_\alpha) \quad \text{for} \quad l = 1, 2, \ldots \text{[*]}$$

Proof. Already the fact that the subgroups A_α generate G implies the equation ${}^1G = (G, G) = \Pi_\alpha {}^1A_\alpha \cdot \overline{(A_\alpha)}$, and that because elements from different A_α commute mod $\overline{(A_\alpha)}$. Now, by Lemma 5.3, Ch. I,

$${}^lG = {}_{l-1}({}^1G) = {}_{l-1}\left[\prod_\alpha {}^1A_\alpha \cdot (\overline{A}_\alpha)\right] = \prod_\alpha {}_{l-1}[{}^1A_\alpha] \cdot {}_{l-1}(A_\alpha),$$

and, by 4.8,

$${}_{l-1}[{}^1A_\alpha]_G = {}_{l-1}(A_\alpha,\ A_\alpha)_G \subseteq {}_{l-1}(A_\alpha,\ A_\alpha)_{A_\alpha} \cdot {}_{l-1}(A_\alpha) = {}^lA_\alpha \cdot {}_{l-1}(A_\alpha).$$

Consequently, ${}^lG \subseteq \Pi_\alpha {}^lA_\alpha \cdot {}_{l-1}(A_\alpha)$. The reverse inclusion is obvious, so that our proof is complete.

6.9. Corollary. *If $G = \Pi^{(k)}_\alpha A_\alpha$, then*

$${}^lG = \prod_\alpha {}^l A_\alpha \quad \text{for} \quad l > k.$$

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[*] Here, as well as in the case when it is followed by normal divisors, the symbol Π denotes all finite products of elements from different A_α. The order of the factors is irrelevant in view of the fact that the supplementary factor ${}_{i-1}(A_\alpha)$ is a normal divisor.

6.10. Theorem. *The k-th nilpotent product* $G = \Pi_\alpha^{(k)} A_\alpha$ *of nilpotent groups* A_α *of class not exceeding* l *is a nilpotent group of class not exceeding* m, *where* $m = \max(k, l)$.

Proof. According to Theorem 6.8, ${}^iG = \Pi^i_\alpha A_\alpha \cdot {}_{i-1}(A_\alpha)$ $(i = 1, 2, \cdots)$. In particular,

$${}^{m+1}G = \prod_\alpha{}^{m+1} A_\alpha \cdot {}_m(A_\alpha) = 1,$$

since $m \geq l$ and $m \geq k$ and, according to 3.5, ${}_k(A_\alpha)^{(k)} = 1$.

6.11. Theorem. *The k-th nilpotent product* G *of nilpotent groups* A_α *of class not exceeding* k *is a maximal nilpotent group of class not exceeding* k *and generated by a system of subgroups respectively isomorphic to the groups* A_α; *every other group which has the same properties is its factor group.*

Proof. The fact that the group G is nilpotent and of class not higher than k follows from Theorem 6.10. Let the group G' be also nilpotent and of class not higher than k and let it be generated by subgroups A'_α respectively isomorphic to the subgroups A_α. Both groups G and G' are factor groups of the free product $F = \Pi^*_\alpha A_\alpha$ with respect to some normal divisors N and N'. By the definition of the k-th nilpotent product, $N = {}_k(A_\alpha)_F$. The nilpotency property of the group G' and the fact that it belongs to a class not higher than the k-th class imply that ${}_k(A'_\alpha)_{G'} = 1$, and therefore (see 2.6, Ch. I) $N' \supseteq N$. This completes the proof of the theorem.

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METABELIAN PRODUCTS OF GROUPS

O. N. GOLOVIN

This paper was written while the author was working on the problem of existence of associative operations on a set of groups other than the direct product and the free product. This problem, which was first proposed by A. G. Kuroš ([1], p. 350), is taken up by the author in papers [3] and [4] in which he constructs denumerably many such operations under the name of nilpotent products (in paper [3] these operations are called central products). The present paper is devoted to a more thorough study of the first nilpotent product which will be referred to, for the sake of brevity and by analogy with nilpotent groups of the corresponding class, as the *metabelian product*. These products were studied before by Levi [2] who referred to them as S-products. In the present paper we have pointed out all items which overlap portions of Levi's paper [2]. In this connection it should be pointed out, however, that we are familiar only with a review of Levi's paper.

It appears that Levi was not at all interested in the problem of associativity of the S-product of groups since he restricted himself to the investigation of the product of two groups only.

The reading of this paper requires familiarity with paper [4]. This is particularly true of the terminology and the notations. The only change in notation pertains to the manner of denoting the metabelian products themselves. Here such products are denoted as follows:

$$A \circ B \ \text{ or } \ \Pi_\alpha^\circ A_\alpha,$$

whereas in [4] the general notations used in connection with nilpotent products made it necessary to denote metabelian products as follows: $A\,(1)\,B$ and $\Pi_\alpha^{(1)} A_\alpha$. The proofs of the basic results of this paper are not based on the proofs of the corresponding results in [4]. In particular, the associativity of metabelian products is proved here in a manner entirely different from that employed in [4]. In deriving the general properties of metabelian products, we do not employ a theorem by Baer on which the corresponding results of [4] were based.

The main result of the present paper consists in the determination of a constructive method of formation of metabelian products of groups belonging to a rather large class of groups, namely, the class of all groups whose commutator factor groups are decomposable into direct products of cyclic groups. This constructive method was in part used by Levi in [2].

§1. Metabelian commutators.

1.1. By the *first nilpotent* or *metabelian product* of the groups A_α, $\alpha \in \mathfrak{M}$ (see

3.2, Ch. II, [4]) we mean the factor group of their free product $F = \prod_\alpha^* A_\alpha$ by ${}_1(A_\alpha)_F = (F, (A_\alpha))$:

$$\prod_\alpha^\circ A_\alpha = \prod_\alpha^* A_\alpha / (F, (A_\alpha)).$$

1.2. The commutator subgroup of the groups A_α in $\prod_\alpha^\circ A_\alpha$, i.e.,

$$(A_\alpha)_Z = [(A_\alpha)\cdot(F, (A_\alpha))] / (F, (A_\alpha)),$$

will be called for brevity the *metabelian commutator subgroup* of the groups A_α.

1.3. It follows from the definition of the metabelian commutator subgroup (cf. 3.2, Ch. II, [4]) that *the metabelian commutator subgroup* $(A_\alpha)_Z$ *of the groups* A_α *is contained in the center of the product* $\prod_\alpha^\circ A_\alpha$ and is therefore a normal divisor in this product (cf. [4], Ch. I, 4.2) as well as an abelian group.

1.4. If $G = \prod_\alpha^\circ A_\alpha$ and Z denotes the center of the group G, then, in view of the inclusion $(A_\alpha)_Z \subset Z$ established in 1.3 above, the basic relations for commutators ([4], Ch. I, 2.1) take on the following simple form (here $a_\alpha, a'_\alpha \in A_\alpha, a_\beta, a'_\beta \in A_\beta, \alpha \neq \beta$):

1.4.1 $(a_\alpha, a_\beta) = (a_\beta, a_\alpha)^{-1}$.

1.4.2. $(a_\alpha a_\alpha', a_\beta) = (a_\alpha, a_\beta)\ (a_\alpha', a_\beta)$;
$(a_\alpha, a_\beta a_\beta') = (a_\alpha, a_\beta)\ (a_\alpha, a_\beta')$.

1.4.3. $((a_\alpha, a_\alpha'), a_\beta) = 1$;
$(a_\alpha, (a_\beta, a_\beta')) = 1$.

1.5. The relations 1.4.2 imply that in the case under consideration the operation of commutation as applied to elements from different factors A_α is distributive with respect to multiplication of elements. The following relations are consequences and special cases of these equations:

1.5.1. $(a_\alpha^k, a_\beta) = (a_\alpha, a_\beta^k) = (a_\alpha, a_\beta)^k$.

1.5.2. $(a_\alpha, a_\beta) = (a_\beta, a_\alpha^{-1}) = (a_\beta^{-1}, a_\alpha) = (a_\alpha^{-1}, a_\beta^{-1})$.

1.5.3. $(a_\alpha, a_\beta)^{-1} = (a_\beta, a_\alpha) = (a_\alpha, a_\beta^{-1}) = (a_\alpha^{-1}, a_\beta) = (a_\beta^{-1}, a_\alpha^{-1})$.

1.6. **Theorem.** *If* $G = \prod_\alpha^\circ A_\alpha$ *and* $a_\alpha \in A_\alpha, a_\beta \in A_\beta, \alpha \neq \beta$, *then the order of the commutator* (a_α, a_β) *divides the orders of the elements* a_α *and* a_β. *(Infinite order is denoted by the number 0.)*

This assertion is a direct consequence of 1.5.1.

§2. On the orders of elements of a metabelian product of groups.

2.1. Since a metabelian product $G = \prod_{\alpha \in \mathfrak{M}}^\circ A_\alpha$ is a regular product, every element g in G has (for some fixed ordering of the set $\mathfrak{M}$ of indices) a uniquely determined regular representation

$$g = a_{\alpha_1} a_{\alpha_2} \dots a_{\alpha_n} u,$$

where $a_{\alpha_i} \in A_{\alpha_i}$, $a_1 < a_2 < \dots < a_n$, $u \in (A_\alpha)_Z$.

Since the commutator subgroup $(A_\alpha)_Z$ is contained in the center Z of the group G, it is possible to give an explicit representation of the product of two elements g and g' in G. Indeed, if $g = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n} u$ and $g' = a'_{\alpha_1} a'_{\alpha_2} \cdots a'_{\alpha_n} u'$ (it is clear that the assumed similarity of the regular representations of g and g' is permissible), then

$$gg' = (a_{\alpha_1} a'_{\alpha_1})\ (a_{\alpha_2} a'_{\alpha_2})\ \dots\ (a_{\alpha_n} a'_{\alpha_n})\, v,$$

where

$$v = uu' \prod_{1\leq i<j\leq n} (a_{\alpha_j}, a'_{\alpha_i}).$$

In particular, the powers of the element $g = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n} u$ are given by the formula:

$$g^m = a_{\alpha_1}{}^m a_{\alpha_2}{}^m \dots a_{\alpha_n}{}^m \cdot u^m \Big(\prod_{1\leq i<j\leq n} (a_{\alpha_j}, a_{\alpha_i})\Big)^{\frac{m(m-1)}{2}}, \qquad (2.1)$$

which holds for an arbitrary integer m.

2.2. Theorem. *Let* $g = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n} u$ *be the regular representation of an element* g *of the metabelian product* $G = \prod^{\circ}_{\alpha} A_\alpha$. *The order* h *of the element* g *is infinite if and only if the order of at least one of the components* a_{α_i} *or* u *is infinite. If, on the other hand, the orders of the components are finite and are given, respectively, by* $s_1 = 2^{l_1} r_1$, $s_2 = 2^{l_2} r_2$, $\dots$, $s_n = 2^{l_n} r_n$, $s = 2^l r$, *where all the numbers* $r_1, r_2, \dots, r_n, r$ *are odd, then, denoting the least common multiple of the orders* $s_1, s_2, \dots, s_n, s$ *by* m, *we have:*

I. $h = m$ *in three cases:*

a) *when* $l_1 = l_2 = \dots = l_n = 0$;

b) *when a single one of the numbers* l_i *is greater than all the other* l_i;

c) *when* $l > 1 + \max(l_1, l_2, \dots, l_n)$.

II. $h = m$ *or (possibly)* $m/2$:

d) *when at least two of the* l_i *are equal, greater than the other* l_i *and positive, and* $l = 1 + \max(l_1, l_2, \dots, l_n)$.

III. $h = m$ *or* $2m$:

e) *when at least two of the* l_i *are equal, greater than the other* l_i *and positive, and* $l < 1 + \max(l_1, l_2, \dots, l_n)$.

Proof. If the order of one of the regular components a_{α_i} is infinite, then the assertion of the theorem is obvious. If the order of each a_{α_i} is finite and the order of the supplementary component u is infinite, then, in view of the fact that the order of each commutator $(a_{\alpha_i}, a_{\alpha_j})$ is finite (see Theorem 1.6), the order of the element g must be infinite (see formula 2.1). Finally, if the orders of all the components including u are finite, then the order of g is obviously finite.

Let us now consider the case when g is of finite order in greater detail. Let m' denote the least common multiple of the numbers $s_1, s_2, \cdots, s_n$. By formula (2.1)

$$g^{m'} = u^{m'} \Big(\prod_{1\leqslant i<j\leqslant n} (a_{\alpha_j}, a_{\alpha_i}) \Big)^{\frac{m'(m'-1)}{2}} = u^{m'} v^{\frac{m'(m'-1)}{2}}, \qquad (*)$$

where all the factors are contained in the abelian group $(A_\alpha)_Z$. In case a) the number m' is odd. On the basis of Theorem 1.6 it follows from the relation (*) that $g^{m'} = u^{m'}$. Consequently, the order of g is m. The same considerations obtain in case b).

It is clear that it suffices to consider case c) for odd m'. Since the order of $u^{m'}$ is a multiple of 4 and the square of the element $v^{m'(m'-1)/2} = v^{-m'/2}$ is certainly (by 1.6) equal to 1, it follows that $g^{2m'} = u^{2m'}$. Consequently, $h = m$.

Now, consider case d). If $v^{m'(m'-1)/2} = 1$, then, as before, $h = m$. If $v^{m'(m'-1)/2} \neq 1$, then the order of the element $v^{m'(m'-1)/2}$ is 2. As for the order of the element $u^{m'}$ it is equal to $m/m' = 2r'$, where r' is odd. It follows that the order of $g^{m'}$ is either r' (this is the case if $u^{m'r'} = v^{m'(m'-1)/2}$), in which case $h = m/2$, or $2r'$, in which case $h = m$.

Finally, consider case e). In that case $u^{m'}$ is of odd order. Consequently, if $v^{m'(m'-1)/2} = 1$, then $h = m$. On the other hand, if the order of the element $v^{m'(m'-1)/2}$ is equal to 2, then $h = 2m$.

2.2.1. **Remark.** It is not difficult to construct examples which show that under the conditions e) h can actually assume either one of the values m and $2m$. Thus, if $G = \{a\} \circ \{b\}$, where $a^4 = b^2 = 1$, then (by 1.6) $(a, b)^2 = 1$, so that the element $a^2 b$ is of order 2 (the case $h = m$). On the other hand, if $G = \{a\} \circ \{b\}$, where $a^2 = b^2 = 1$, then, as will be shown below (see 4.3.1), G is a well-known non-abelian group of order 8 different from the group of quaternions. In this group the element ab is of order 4.

Case d) with $h = m$ can be illustrated using the product $G = \{a\} \circ \{b\}$, where $a^4 = b^4 = 1$. The element $a^2 b^2 (a, b)$ in that product satisfies the conditions formulated in d) (see 4.3.3 below) and its order is 4 (i.e., $h = m$). As regards the possible value $h = m/2$, the question remains an open one.

§3. $\mathfrak{W}^*$-ϕ-products of groups.

3.1. Let there be given an arbitrary set of groups A_α, where the index varies over a certain ordered set $\mathfrak{M}$, and, in addition, a group $\mathfrak{W}^*$. We consider an arbitrary "homomorphic mapping" ϕ of the set of groups A_α into the center of the group $\mathfrak{W}^*$, i.e., a correspondence which associates with every pair of elements a_α, a_β ($a_\alpha \in A_\alpha$, $a_\beta \in A_\beta$), where α and β vary over the whole set $\mathfrak{M}$ and $\alpha < \beta$, a definite element in the center of the group $\mathfrak{W}^*$, which element is denoted by (a_α, a_β). It is assumed that the following relations hold:

$$\begin{aligned}(a_\alpha a_\alpha', a_\beta) &= (a_\alpha, a_\beta)(a_\alpha', a_\beta),\\ (a_\alpha, a_\beta a_\beta') &= (a_\alpha, a_\beta)(a_\alpha, a_\beta').\end{aligned} \tag{3.1}$$

For any set of groups A_α and for any group $\mathfrak{W}^*$ there exists at least one such homomorphic mapping. Indeed, the relations (3.1) are trivially satisfied if we associate with every pair of elements a_α, a_β the identity of the group $\mathfrak{W}^*$.

We call the element (a_α, a_β) *the* $\mathfrak{W}^*$*-commutator* of the elements a_α and a_β ($\alpha < \beta$). As the $\mathfrak{W}^*$-commutator of the elements a_β and a_α (again with $\alpha < \beta$) we define the element $(a_\beta, a_\alpha) = (a_\alpha, a_\beta)^{-1}$. The equalities $(a_\alpha, a_\beta) = (1 \cdot a_\alpha, a_\beta) = (1, a_\beta)(a_\alpha, a_\beta)$ imply that $(1, a_\beta) = 1$. Likewise, $(a_\alpha, 1) = 1$. The relations (3.1) coincide with the relations 1.4.2. Hence, the relations (3.1) imply the relations 1.5.1–1.5.3 which are formal consequences of the relations 1.4.2.

3.2. Definition. By the $\mathfrak{W}^*$-ϕ-product* of a set of groups A_α, where α varies over an ordered set $\mathfrak{M}$ and ϕ is a certain homomorphic mapping of the groups A_α into the center of the group $\mathfrak{W}^*$, we mean the set of all finite words of the form

$$g = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n} w,$$

where $a_{\alpha_i} \in A_{\alpha_i}$ ($\alpha_1 < \alpha_2 < \cdots < \alpha_n$), $w \in \mathfrak{W}^*$ (with words differing from one another by "basic components" a_{α_i} equal to 1 identified) with the following definition of multiplication:

$$(a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_n} w)(a_{\alpha_1}' a_{\alpha_2}' \ldots a_{\alpha_n}' w') = (a_{\alpha_1} a_{\alpha_1}')(a_{\alpha_2} a_{\alpha_2}') \ldots (a_{\alpha_n} a_{\alpha_n}') w'',$$

where $w'' = ww' \prod_{1 \le j < i \le n} (a_{\alpha_i}, a'_{\alpha_j})$.

3.3. Theorem. *The* $\mathfrak{W}^*$-ϕ-*product* G *of a set of groups* A_α *is a group.*

Proof. The operation defined in G is associative. Indeed, consider in G three arbitrary elements $g_i = a^{(i)}_{\alpha_1} a^{(i)}_{\alpha_2} \cdots a^{(i)}_{\alpha_n} w^{(i)}$, $i = 1, 2, 3$. It is obvious that the basic components in the products $(g_1 g_2) g_3$ and $g_1(g_2 g_3)$ coincide. The $\mathfrak{W}^*$ components of these products also coincide for, in view of equations (3.1), those components are equal, in either product, to

$$w^{(1)} w^{(2)} w^{(3)} \prod_{1 \leqslant k < l \leqslant 3} \prod_{1 \leqslant j < i \leqslant n} (a_{\alpha_i}{}^{(k)}, a_{\alpha_j}{}^{(l)}).$$

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* Cf. Levi [2].

The unit element is the word all of whose components are 1. The inverse element of the element $g = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n} w$ is the element

$$g^{-1} = a_{\alpha_1}{}^{-1} a_{\alpha_2}{}^{-1} \dots a_{\alpha_n}{}^{-1} \left(w^{-1} \prod_{1 \leqslant j < l \leqslant n} (a_{\alpha_l}, a_{\alpha_j}) \right).$$

3.4. Remark. The W^*-ϕ-product of G of the groups A_α contains subgroups isomorphic to the groups W^* and A_α, respectively, and is generated by those groups. We may, therefore, say that *the* W^*-ϕ-*product of the groups* A_α *is generated by the groups* W^* *and* A_α, $\alpha \in \mathfrak{M}$, *themselves*. Also, it is clear that the intersection of any two of these subgroups of G consists of the unit element only. In addition, W^* is a normal divisor of the group G, as is seen from the fact that W^* is contained in the centralizor of the set of all elements of the groups A_α, $\alpha \in \mathfrak{M}$, and the factor group G/W^* is isomorphic to the direct product $\Pi^{\times}_{\alpha} A_\alpha$.

3.5. Remark. Since the commutator subgroup $(A_\alpha)_G$ is contained in the center of the subgroup W^* and W^* is contained in the centralizor of the subgroup $\{A_\alpha\}$, we conclude that the commutator subgroup $(A_\alpha)_G$ is contained in the center of the group G. Consequently, the relations 1.4.3 hold for the commutators (a_α, a_β).

3.6. Remark. In the sequel we consider only those W-ϕ-products G of groups A_α, $\alpha \in \mathfrak{M}$, for which $(A_\alpha) = W$ (this means, among others, that W must be an abelian group). Such W-ϕ-products will be denoted by $\Pi^{W}_{\alpha} A_\alpha$. We observe that while this notation does not contain a reference to the homomorphism ϕ, the product $\Pi^{W}_{\alpha} A_\alpha$ is not defined unless both the group W and the homomorphism ϕ are given. A group W with which it is possible to construct the product $\Pi^{W}_{\alpha} A_\alpha$ will be said to be *admissible* for the groups A_α.

It is clear that every W^*-ϕ-product G of the groups A_α contains the product $\Pi^{W}_{\alpha} A_\alpha$, where $W = (A_\alpha)_G$.

3.7. Theorem. *A group G is a product $\Pi^{W}_{\alpha} A_\alpha$ if and only if it is a regular factor group of the metabelian product $H = \Pi^{\circ}_{\alpha} A_\alpha$ of those very same groups.*

Proof. Let $G = \Pi^{W}_{\alpha} A_\alpha$. By Theorem 1.2, Ch. II, [4], G is a regular product of the groups A_α. Since $(A_\alpha)_G$ is contained in the center of the group G, the latter is, by Dyck's theorem ([1], p. 86), a factor group of the metabelian product $H = \Pi^{\circ}_{\alpha} A_\alpha$. The converse is obvious if we put $W = (A_\alpha)_G$ and associate with a pair of elements a_α, a_β $(a_\alpha \in A_\alpha, a_\beta \in A_\beta, \alpha < \beta)$ their commutator (a_α, a_β) in G.

3.8. Corollary. *A group W is admissible for the groups A_α if and only if it is isomorphic to a factor group of the metabelian commutator subgroup* $(A_\alpha)_Z$.

§4. Construction of metabelian products of certain pairs of groups.

4.1. Theorem 3.7 which establishes the "maximality" of the metabelian product $\Pi^{\circ}_{\alpha} A_\alpha$ among all products $\Pi^{W}_{\alpha} A_\alpha$ for all admissible groups W and all homomorphisms ϕ enables us to construct metabelian products for many classes of groups. Such constructions are described in the sequel. Since every product $\Pi_\alpha^W A_\alpha$ of prescribed groups A_α is defined by a group W and a homomorphism ϕ, our purpose is best served by Corollary 3.8 of Theorem 3.7 rather than by Theorem 3.7 itself. The general construction procedure can be outlined as follows.

Let there be given a set of groups A_α. We first determine the properties of groups admissible for the given set of groups A_α. We then construct the "most extensive" admissible group. Finally, we prove rigorously the "maximality" of that admissible group, i.e., we prove that every admissible group is its factor group. This maximal admissible group is at the same time the metabelian commutator subgroup of the given groups A_α.

In this section we consider the simplest case, namely, the case of two groups.

4.2. Theorem. *The metabelian commutator subgroup* $(A, B)_Z$ *of two cyclic groups* $A = \{a\}$, $B = \{b\}$ *of order* k *and* l, *respectively, (infinite order is denoted by the number 0) is a cyclic group* $(A, B)_Z = \{(a, b)\}$ *whose order* n *is equal to the greatest common divisor* d *of the numbers* k *and* l.

Proof. In view of the relation 1.5.1, the group $(A, B)_Z$ is a cyclic group generated by the element (a, b) and its order n is a divisor of the number d. Now, the cyclic group $W = \{w\}$ of order d is admissible for A and B. Indeed, the correspondence ϕ between pairs of elements $[a^i, b^j]$ and elements in W defined by the equation $[a^i, b^j]\phi = w^{ij}$ satisfies the relations (3.1):

$$[a^i a^{i'}, b^j]\varphi = [a^{i+i'}, b^j]\varphi = w^{(i+i')j} = w^{ij}w^{i'j} = [a^i, b^j]\varphi\cdot[a^{i'}, b^j]\varphi$$

and, analogously, $[a^i, b^j b^{j'}]\varphi = [a^i, b^j]\varphi\cdot[a^i, b^{j'}]\varphi$. Consequently, by 3.8, $n = d$, which proves our theorem.

4.3. Corollary. *The order of the metabelian product* $G = \{a\} \circ \{b\}$ *of two cyclic groups of order* k *and* l, *respectively, is equal to* kld *where* $d = (k, l)$.

Indeed, the order of a metabelian product and, more generally, of any regular product is equal to the product of the orders of the factors and their commutator subgroup (see [4], Ch. II, 1.2).

This corollary allows us to draw a number of particular conclusions:

4.3.1. The metabelian product $G = \{a\} \circ \{b\}$ of two cyclic groups of order *2* is a well-known non-abelian group of order *8* different from the group of quaternions. Indeed, in accordance with 4.3, the order G is *8*; G cannot be the group of quaternions since its subgroups $\{a\}$ and $\{b\}$ are not normal divisors (see [4], Ch. II, 1.9).

4.3.2. The metabelian product $G = \{a\} \circ \{b\}$ of two cyclic groups of prime order p $(p > 2)$ is a well-known non-abelian group of order p^3 and the orders of its elements are equal to p (see 2.2).

4.3.3. The metabelian commutator subgroup of two cyclic groups of equal order k is a cyclic group of order k, and their metabelian product is a non-abelian group of order k^3.

4.3.4. The metabelian commutator subgroup of two infinite cyclic groups is an infinite cyclic group and their metabelian product is a non-abelian group without torsion (i.e., a group in which 1 is the only element of finite order).

4.4. Theorem. *The metabelian commutator subgroup* $(A, B)_Z$ *of an infinite cyclic group* $A = \{a\}$ *and an arbitrary abelian group* B *is isomorphic to the group* B.

Proof. In view of 1.4.2 the metabelian commutator subgroup $(A, B)_Z$ is equal to $(a, B)_Z$ and is, therefore, a factor group of the group B. Now, B is admissible for the groups A and B since the mapping ϕ of the pair of groups A and B into B defined by the equation $[a^i, b]\phi = b^i$ satisfies the conditions (3.1):

$$[a^{i+j}, b]\varphi = b^{i+j} = b^i b^j = [a^i, b]\varphi\cdot[a^j, b]\varphi$$

and

$$[a^i, bb']\varphi = (bb')^i = b^i b'^i = [a^i, b]\varphi\cdot[a^i, b']\varphi.$$

In addition, ϕ is a mapping onto the whole group B. The latter conclusion follows from the fact that $[a, b]\phi = b$.

4.4.1. Remark. It is clear that Theorem 4.4 admits of the following generalization:

The metableian commutator subgroup $(A, B)_Z$ of a cyclic group $A = \{a\}$ of order k and an arbitrary abelian group B is isomorphic to the factor group B/B^k, where B^k is the subgroup of the group B generated by the k-th powers of all of its elements.

4.4.2. Corollary. *Given an abelian group there exists a (metabelian) group whose commutator subgroup is isomorphic to the given group.*

Indeed, such a group is the metabelian product of the given group and an infinite cyclic group.

4.5. Theorem.* *If* $G = A \circ B$ *and* A *and* B *are decomposed into direct products:* $A = \Pi^{\times}_{\alpha} A_{\alpha}$, $B = \Pi^{\times}_{\beta} B_{\beta}$, *then, for an arbitrary pair of indices* α *and* β, $(A_{\alpha}, B_{\beta})_G \cong (A_{\alpha}, B_{\beta})_Z$ *and* $(A, B)_Z = \Pi^{\times}_{\alpha,\beta} (A_{\alpha}, B_{\beta})_G$.

Proof. In view of 1.4.2, the commutator subgroup $(A, B)_Z$ is generated by the commutator subgroups $(A_{\alpha}, A_{\beta})_G$, where α and β assume all their values. By Theorem 6.1, Ch. II, [4], the commutator subgroup $(A_{\alpha}, A_{\beta})_G$ the factor group of the the metabelian commutator subgroup of these very same groups. Consequently, the commutator subgroup $(A, B)_Z$ is a factor group of the group $W = \Pi^{\times}_{\alpha,\beta} (A_\alpha, A_\beta)_Z$. Now, the group W is admissible for the groups A and B. Indeed, let us associate with each pair of elements $a = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n}$ in A and $b = b_{\beta_1} b_{\beta_2} \cdots b_{\beta_m}$ in B the element

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* In its simplest form this theorem was proved by Levi [2].

$$\prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (a_{\alpha_i}, b_{\beta_j})z$$

in W. This correspondence satisfies (3.1) since

$$\prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (a_{\alpha_i} a_{\alpha_i}', \ b_{\beta_j})z = \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (a_{\alpha_i}, \ b_{\beta_j})z \cdot \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (a_{\alpha_i}', \ b_{\beta_j})z$$

and, analogously,

$$\prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (a_{\alpha_i}, \ b_{\beta_j} b_{\beta_j}')z = \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (a_{\alpha_i}, \ b_{\beta_j})z \cdot \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (a_{\alpha_i}, \ b_{\beta_j}')z.$$

To complete the proof it suffices to observe that in the W-product of the groups A and B the commutator subgroup of any pair of subgroups A_α, B_β is $(A_\alpha, B_\beta)_Z$.

4.5.1. Corollary. Since we know the nature of the metabelian commutator subgroup of two arbitrary cyclic groups (see 4.3), Theorem 4.5 permits us to construct the metabelian commutator subgroup (and so the metabelian product) of two arbitrary abelian groups each of which can be decomposed into a direct product of cyclic subgroups. In particular, this theorem permits us to construct the metabelian commutator subgroup (and so the metabelian product) of two arbitrary abelian groups with a finite number of generators.

4.6. Lemma. *If $G = A \circ B$, then every element in one of these groups commutes with every element in the commutator subgroup of the other group.*

Indeed, in case of the group G, equations 1.4.3 become: $((a, a'), b) = 1$, $(a, (b, b')) = 1$, where $a, a' \in A$, $b, b' \in B$. But these relations imply that every commutator of the subgroup A, say, commutes with every element in the subgroup B, i.e., $((A, A), B)_G = 1$. Similarly, $(A, (B, B))_G = 1$.

It is clear that this lemma is a special case of Corollary 4.6, Ch. II, [4].

4.7. Theorem. *The metabelian commutator subgroup $(A, B)_Z$ of two groups A and B is isomorphic to the metabelian commutator subgroup $(A^*, B^*)_Z$ of their factor groups by their commutator subgroups: $A^* = A/(A, A)$, $B^* = B/(B, B)$.*

Proof. Let $G = A \circ B$. We shall show that $(A, B)_G$ is an admissible group for A^* and B^*. If we put $a^* = a(A, A)$ and $b^* = b(B, B)$, then, in view of 4.6, the mapping $[a^*, b^*]\phi = (a, b)$ is well-defined. Also, it is easy to see that this mapping is a homomorphic mapping in the sense of 3.1. Consequently, $(A, B)_G$ is indeed an admissible group for A^* and B^*.

If now W is an arbitrary admissible group for A^* and B^*, it is also admissible for the groups A and B; this in view of the following obvious general assertion:

If a group is admissible for factor groups of the groups A_α, $\alpha \in \mathfrak{M}$, then it is admissible for the groups A_α themselves.

Consequently, W is isomorphic to a factor group of the group $(A, B)_G$ (see 3.8), which proves the theorem.

4.8. Theorem. *If $G = A \circ B$ and A and B are decomposed into arbitrary regular products: $A = \Pi_\alpha^\Delta A_\alpha$, $B = \Pi_\beta^\Delta B_\beta$, then $(A_\alpha, B_\beta)_G \cong (A_\alpha, B_\beta)_Z$ for an arbitrary pair of indices α, β and the metabelian commutator subgroup $(A, B)_G$ is decomposable into the direct product $\Pi_{\alpha,\beta}^\times (A_\alpha, B_\beta)_G$.*

This theorem is a generalization of Theorem 4.5 and its proof is, in view of 4.7, an immediate consequence of that theorem.

§5. Construction of metabelian products of arbitrary sets of groups belonging to certain classes.

5.1. Theorem. *The metabelian commutator subgroup $(A_\alpha)_Z$ of an arbitrary set of groups A_α, $\alpha \in \mathfrak{M}$, can be decomposed into the direct product of the metabelian commutator subgroups of pairs of groups A_α:*

$$(A_\alpha)_Z = \prod_{\alpha \neq \beta}^{\times} (A_\alpha, A_\beta)_Z.$$

Proof. By definition (see [4], Ch. I, 1.3), $(A_\alpha)_Z = \{(A_\alpha, A_\beta)_G, \alpha \neq \beta\}$, where $G = \Pi_\alpha^\circ A_\alpha$. The remainder of the proof is entirely analogous to the proof of Theorem 4.5. If we do not make use of the general Theorem 5.5, Ch. II, [4] concerning the associativity of nilpotent products, we cannot, at first, claim that $(A_\alpha, A_\beta)_G \cong (A_\alpha, A_\beta)_Z$. We can, however, conclude on the basis of Theorem 6.1, Ch. II, [4]* that the commutator subgroup $(A_\alpha, A_\beta)_G$ is isomorphic to a factor group of the metabelian commutator subgroup $(A_\alpha, A_\beta)_Z$. Consequently, the commutator subgroup $(A_\alpha)_Z$ is a factor group of the group $W = \Pi^\times (A_\alpha, A_\beta)_Z$. All that remains to show is that the group W is admissible for the set of groups A_α. A homomorphic mapping ϕ of the set of groups A_α into W is given by the equation: $[a_\alpha, a_\beta]\phi = (a_\alpha, a_\beta)_Z$. Since, by its very definition, a homomorphic mapping is to satisfy the conditions (3.1) only for pairs of subgroups A_α, A_β, these conditions are automatically fulfilled in view of the definition of the mapping ϕ. Thus, W is admissible for the groups A_α, and our theorem is proved.

This theorem enables us to reduce the problem of the construction of the metabelian commutator subgroup, and so of the metabelian product, of an arbitrary set of groups to the case of two groups, which case was considered above. In particular, the following general theorems are immediate consequences of corresponding theorems proved for two groups:

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* It is easily verified that the proof of Theorem 6.1, Ch. II, [4], is not at all connected with the whole system of lemmas in §4, Ch. II, [4], which precedes the proof of the general theorem on associativity.

5.2. Theorem. *The metabelian commutator subgroup* $(A_\alpha)_Z$ *of a set of groups* A_α *each of which is a regular product of its subgroups* $A_\alpha^{\beta_\alpha}$: $A_\alpha = \Pi^{\Delta}_{\beta_\alpha \in \mathfrak{M}_\alpha} A_\alpha^{\beta_\alpha}$ *can be decomposed into a direct product:*

$$(A_\alpha)_Z = \prod_{\alpha \neq \alpha'}^{\times} \prod_{\substack{\beta_\alpha \in \mathfrak{M}_\alpha \\ \beta_{\alpha'} \in \mathfrak{M}_{\alpha'}}}^{\times} (A_\alpha^{\beta_\alpha}, A_{\alpha'}^{\beta_{\alpha'}})_Z.$$

(see Theorem 4.8).

5.3. Theorem. *The metabelian commutator subgroup* $(A_\alpha)_Z$ *of a set of groups* A_α *coincides with the metabelian commutator subgroup* $(A_\alpha^*)_Z$ *of the factor groups of the factors by their commutator subgroups:* $A_\alpha^* = A_\alpha/(A_\alpha, A_\alpha)$ (see Theorem 4.7).

5.4. Remark. While the definition of the metabelian product of groups (see 1.1) as the factor group of their free product by the principal commutator subgroup of their free commutator subgroup enables us (as was shown in §6 of Chapter II, [4]) to derive certain properties of that product, it does not allow us to construct directly the metabelian product of given factors. Nor is this problem solved by Theorem 1.2, Ch. II, [4], which asserts the existence of a unique regular representation of the elements of an arbitrary regular product but fails to clarify the structure of the normalized commutator subgroup of the factors.

Theorems 5.1, 5.2, and 5.3 permit us to say that for a large class of groups it is possible, in principle, to construct their metabelian product. Indeed, since the commutator subgroup $(A_\alpha)_Z$ of the factors of the metabelian product $G = \Pi^\circ A_\alpha$ is contained in the center of that product, it suffices to know the commutator subgroup $(A_\alpha)_Z$ in order to be able to carry out effectively the multiplication of two elements in G (provided we know how to do that in each of the factors A_α). We observe that by "knowing" the commutator subgroup $(A_\alpha)_Z$ we mean knowing it not as an abstractly given group but as a group generated by the commutators (a_α, a_β). Knowledge of the commutator subgroup $(A_\alpha)_Z$ enables us to solve another problem for the elements of G, namely, the identity problem. Theorems 5.1 and 5.3 imply that in order to know the commutator subgroup $(A_\alpha)_Z$ it suffices to know the commutator subgroups $(A_\alpha/(A_\alpha, A_\alpha), A_\beta/(A_\beta, A_\beta))$ whose direct product is equal to $(A_\alpha)_Z$. Consequently, if the groups A_α are such that their factor groups with respect to their commutator subgroup can be decomposed into direct products of cyclic groups and if it is assumed that we can actually construct those factor groups and realize their decomposition into direct products, then, in view of the fact that the metabelian commutator subgroups of pairs of subgroups have been fully described (see

4.2), we may assert that it is also possible to construct the commutator subgroup $(A_\alpha)_Z$.

The most important example of such a "good" class of groups is the class of groups with a finite number of generators. In particular, this class contains all finite groups.

§6. General properties of metabelian products of groups.

In this section we give a new proof of the associativity of metabelian products of groups. This proof does not depend on the proof of the general theorem on the associativity of nilpotent products given by the author in 5.5, Ch. II, [4]. In addition to the proof of associativity we give independent proofs of theorems which are special cases of theorems obtained in §6, Ch. II, [4] for arbitrary nilpotent products and proved there using a certain theorem by Baer.

6.1. **Theorem.** *A regular product* $G = \Pi^{\Delta}_{\alpha} A_\alpha$ *is the metabelian product of the groups* A_α *if and only if the commutator subgroup* $(A_\alpha)_G$ *is contained in the center of the group* G *and can be decomposed into the direct product of the commutator subgroups* $(A_\alpha, A_\beta)_G$ *for all pairs of indices* $\alpha, \beta, \alpha \neq \beta$, *each of which (groups) coincides with the metabelian commutator subgroup* $(A_\alpha, A_\beta)_Z$ *of the groups* A_α *and* A_β.

Indeed, one part of this theorem has already been proved in 5.1. On the other hand, if $G = \Pi^{\Delta}_{\alpha} A_\alpha$ and the commutator subgroup $(A_\alpha)_G = \Pi^{\times}_{\alpha\neq\beta} (A_\alpha, A_\beta)_Z$ is contained in the center of the group G, then, by the first part of this theorem, it is possible to map G isomorphically on the metabelian product $G^* = \Pi^{\Delta}_{\alpha} A_\alpha$ of the groups A_α in such a way that every group A_α will be mapped onto itself and every commutator subgroup $(A_\alpha, A_\beta)_G$ will be mapped onto $(A_\alpha, A_\beta)_Z$. Thus, $G = \Pi^{\Delta}_{\alpha} A_\alpha$.

6.2. **Corollary.** *If* $G = \Pi^{\circ}_{\alpha\in\mathfrak{M}} A_\alpha$ *and* $\mathfrak{N}$ *is an arbitrary subset of* $\mathfrak{M}$, *then* $\{A_\alpha, \alpha \in \mathfrak{N}\} = \Pi^{\circ}_{\alpha\in\mathfrak{N}} A_\alpha$. *In particular,* $\{A_\alpha, A_\beta\} = A_\alpha \circ A_\beta$.

6.3. **Theorem.** *Metabelian multiplication is associative, i.e., if* $G = \Pi^{\circ}_{\gamma\in\mathfrak{M}} G_\gamma$ *and every group* G_γ *is itself decomposable into a metabelian product*

$$G_\gamma = \prod^{\circ}_{\alpha_\gamma\in\mathfrak{N}_\gamma} A_\alpha^{\alpha_\gamma}$$

then G *is the metabelian product:*

$$G = \prod^{\circ}_{\substack{\alpha_\gamma\in\mathfrak{N}_\gamma \\ \gamma\in\mathfrak{M}}} A_\alpha^{\alpha_\gamma}.$$

Proof. According to Theorem 5.2, it is possible to decompose the metabelian commutator subgroup $(G_\gamma, \gamma \in \mathfrak{M})_Z$ into the following direct product:

$$(G_\gamma)_Z = \prod_{\gamma \neq \gamma'}^{\times} \prod_{\substack{\alpha_\gamma \in \mathfrak{N}_\gamma \\ \beta_{\gamma'} \in \mathfrak{N}_{\gamma'}}}^{\times} (A_\gamma^{\alpha_\gamma}, A_{\gamma'}^{\beta_{\gamma'}}).$$

In addition, by 5.1, we have for every fixed γ: $(A_\gamma^{\alpha_\gamma}, \alpha_\gamma \in \mathfrak{N}_\gamma) = \prod^{\times}_{\alpha_\gamma \neq \beta_\gamma} (A_\gamma^{\alpha_\gamma}, A_\gamma^{\beta_\gamma})$. Since $(A_\gamma^{\alpha_\gamma}, \alpha_\gamma \in \mathfrak{N}_\gamma)$ lies in the center of the factor G_γ and at the same time, by 4.6 and 6.2, in the centralizor of every other factor G_δ with $\delta \neq \gamma$, it follows that $(A_\gamma^{\alpha_\gamma}, \alpha \in \mathfrak{N}_\gamma)$ is contained in the center of the group G. In addition, in view of the regularity of the metabelian product $\prod^{\circ}_{\gamma \in \mathfrak{M}} G_\gamma$, the intersection of $(A_\gamma^{\alpha_\gamma}, \alpha_\gamma \in \mathfrak{N}_\gamma)$ and $(G_\gamma, \gamma \in \mathfrak{M})_Z \cdot \prod_{\delta \neq \gamma} G_\delta$ consists of the unit element only. It follows that the commutator subgroup $(G_\gamma, \gamma \in \mathfrak{M})_Z$ multiplies directly with all the commutator subgroups $(A_\gamma^{\alpha_\gamma}, \alpha_\gamma \in \mathfrak{N}_\gamma)$. Consequently, the commutator subgroup $(A_\gamma^{\alpha_\gamma}, \alpha_\gamma \in \mathfrak{N}_\gamma, \gamma \in \mathfrak{M})$ can be decomposed into the direct product of all commutator subgroups $(A_\gamma^{\alpha_\gamma}, A_{\gamma'}^{\beta_{\gamma'}})$ of pairs of groups $A_\gamma^{\alpha_\gamma}$, $A_{\gamma'}^{\beta_{\gamma'}}$ with either $\gamma \neq \gamma'$ or, if $\gamma = \gamma'$, with $\alpha_\gamma \neq \beta_{\gamma'}$. Finally, it is not difficult to see that G is regularly generated by the subgroups $A_\gamma^{\alpha_\gamma}$ (see e.g., [4], Ch. II, 1.2). Hence, by Theorem 6.1, G is the metabelian product of the groups $A_\gamma^{\alpha_\gamma}$.

6.4. In proving Theorems 6.3, 6.4, 6.6, and 6.7 of Ch. II in [4] for the case of a metabelian product it is not necessary to make use of Corollary 5.13, Ch. I, [4], to Baer's theorem. Indeed, the proofs are direct consequences of Theorems 1.6 and 2.2 and of the definition of a metabelian product. All these results are stated in the following

Theorem. *The metabelian product G of locally finite groups is locally finite; the metabelian product of periodic groups is periodic and the characteristic* of G is the sum of the characteristics of the factors. In particular, the metabelian product of p-groups (with the same p) is a p-group and the metabelian product of a finite number of finite groups is a finite group.*

6.5. **Theorem.** *If $G = \prod^{\circ}_{\alpha} A_\alpha$, then the commutator subgroup of G satisfies the the relation*

$$(G, G) = \prod_{\alpha}^{\times} (A_\alpha, A_\alpha) \times (A_\alpha) = \prod_{\alpha, \beta}^{\times} (A_\alpha, A_\beta), \tag{6.5}$$

and all the other terms of the lower central series satisfy the relation

$${}^{l}G = \prod_{\alpha}^{\times} {}^{l}A_\alpha \quad (l > 1).$$

Indeed, the assertion concerning the commutator subgroup follows, in view of Lemma 4.6 and Theorem 5.1 from the regularity of the metabelian product and from the fact that $(A_\alpha)_Z$ is contained in the center of the group G. These theorems also imply the correctness of the expression for ${}^{l}G$.

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* i.e., the set of all prime numbers which appear in the order of the elements of the group. The characteristic includes the number *0* if at least one of the elements of the group is of infinite order (see [4], Ch. II, 6.3).

This theorem is an improvement over the general Theorem 6.8 and Corollary 6.9 of Ch. II [4] as applied to the special case of the first nilpotent product. Just as in the general case, a direct consequence of the theorem just proved is

6.6. **Theorem.** *The metabelian product G of nilpotent groups of class $k \leq l$,* of which at least one is of class l is a nilpotent group. If $l > 0$, then the product is of class l. If $l = 0$, then the product is either of class 0 or of class 1.*

That both cases mentioned in the theorem can take place for $l = 0$ is apparent from examples of metabelian products of two finite cyclic groups. Indeed, as was proved in 4.3, the metabelian product of two finite cyclic groups whose orders are not co-prime is not abelian. On the other hand, if these orders are co-prime, the commutator subgroup of the product is 1.

6.7. We conclude this paper with a remark concerning a problem of fundamental interest.

In remark 6.1.1, Ch. II of [4], it was pointed out that if $G = \Pi_{a}^{(k)} A_{a}$ and if, for every value of a, B_{a} is a subgroup in A_{a}, then $H = \{B_{a}\}$ is a factor group of the k-th nilpotent product $\Pi_{a}^{(k)} B_{a}$ of the groups B_{a} (see Theorem 6.1, Ch. II [4]) which, in general, is not isomorphic to that product; this in distinction to the relations which obtain in case of the direct product and the free product of groups. It is now easy to give an example which illustrates this difference.

Let $G = \{a\} \circ \{b\}$ where both elements a and b are of order 4. By 4.3.3, the commutator subgroup $(\{a\}, \{b\}) = \{(a, b)\}$ is also a cyclic group of order 4. The subgroups $\{a^2\}$ and $\{b^2\}$ contained in the respective factors of the product under consideration are both cyclic groups of order 2 and their commutator subgroup in G consists of the unit element only (for, $(a^2, b^2) = (a, b)^4 = 1$.) Hence, $\{a^2, b^2\} = \{a^2\} \times \{b^2\}$. On the other hand, by 4.3.1, the metabelian product of two cyclic groups of order 2 is a non-abelian group of order 8.

The above essential difference between nilpotent products on the one hand and free and direct products on the other made it possible for A. I. Mal'cev to pose the problem of the existence of fully regular operations (see [4]), Ch. II, 2.1) on sets of groups, different from the direct and free product operations and having the additional property that multiplication of arbitrary subgroups of groups of the set is analogous to the multiplication of the groups themselves.

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* A group G is called *nilpotent* if its lower central series reaches the unit element after a finite number of terms. A nilpotent group is said to be *of class l* if ${}^{l}G \neq 1$ and ${}^{l+1}G = 1$.

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ON THE PROBLEM OF ISOMORPHISMS OF NILPOTENT DECOMPOSITIONS OF A GROUP

O. N. GOLOVIN

In his papers [3], [4], and [5], the author introduced and investigated nilpotent products of groups. (For a definition of nilpotent products see [4], Ch. II, 3.2.) In this paper we pose the problem of determining the connections among the possible nilpotent decompositions of a group. This problem can be subdivided into the following two problems:

A) What can be said about two nilpotent decompositions with the same k of a group G:

$$G = \prod_{\alpha}^{(k)} A_\alpha = \prod_{\beta}^{(k)} B_\beta?$$

B) Is it possible to decompose a group G into two nilpotent products

$$G = \prod_{\alpha}^{(k)} A_\alpha = \prod_{\beta}^{(l)} B_\beta,$$

where $k \neq l$? If the latter question can be answered in the affirmative what connections exist between two such decompositions?

Finally, one case raises the following question:

C) Is it possible to decompose a given group into a free product as well as into some nilpotent product?

The importance of problem A) in the literature on groups in the special case of direct decompositions of a group (which are 0-th nilpotent decompositions) is well known. The same can be said about the difficulties besetting the solution of this problem for sufficiently large classes of groups. We may therefore limit ourselves, at first, to some small class of groups, the class of finite groups, say, and ask whether two indecomposable nilpotent decompositions with the same k of a group in this class are isomorphic (in the usual sense of it being possible to establish a one-to-one correspondence between the factors of the two decompositions under which corresponding factors are isomorphic groups). In this paper a first step is made in that direction; namely, we prove (see 2.1) that if a finite group can be decomposed into a k-th nilpotent product of cyclic groups, then all its indecomposable k-th nilpotent products are isomorphic.

As for B), we prove (see 1.4) that it is possible for a decomposition to be a direct decomposition and at the same time every one of the nilpotent decompositions.

C) is fully solved by an elementary extension to arbitrary nilpotent products of a well-known result of Baer and Levi (see, e.g., [1], p. 350) which asserts that if a group can be decomposed into a direct product of its proper subgroups, then it cannot be decomposed into a free product.

In this paper we often refrain from giving the definitions of specialized concepts. Instead, the reader is referred to the appropriate portions of papers [4] and [5].

§1. Degeneration of nilpotent decompositions of groups.

1.1. Definition. The k-th nilpotent product (see [4], Ch. II, 3.2) $G = \Pi_{\alpha}^{(k)} A_{\alpha}$ is said *to degenerate into an l-th* ($l < k$) nilpotent product if the group G is also the l-th nilpotent product of the groups A_{α}. Two such decompositions are said to be *coincident.*

It is clear that degeneration takes place if and only if the k-th free commutator subgroup of the groups A_{α} coincides with the l-th free commutator subgroup of those groups.

1.2. Lemma. *The coincidence of the k-th nilpotent product of the groups A_{α} with the l-th nilpotent product of these groups ($l < k$) implies the degeneration into the l-th nilpotent product of all nilpotent products of the groups A_{α} whose numbers* exceed l.*

This assertion is an immediate consequence of the fact that coincidence of two terms of any minimal central series (see [4], Ch. I, 5.1) implies coincidence with these two terms of all terms of the series which follow at least one of them.

1.3. Corollary. *If the first nilpotent product of the groups A_{α} degenerates into their direct product, then all nilpotent products of these groups degenerate into their direct product.*

1.4. By the *characteristic* of a group G we shall mean the totality of prime numbers which are divisors of the orders of the elements of the group. If at least one of the elements of the group is of infinite order, then the characteristic includes the number 0. We shall call the characteristics of two groups *co-prime* if they have only the number 1 in common.

As usual, a group G will be called *complete* if it contains a solution of the equation $x^n = g$ for every integer n and every element $g \in G$.

Theorem. *Degeneration of all the nilpotent products of the groups A_{α} into their direct product takes place if:*

1) *the groups $A_{\alpha}/(A_{\alpha}, A_{\alpha})$ are periodic and their characteristics are pairwise co-prime;*

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* Translator's note: the number of $\Pi_{\alpha}^{(k)} A_{\alpha}$ is k.

2) *the groups* $A_\alpha/(A_\alpha, A_\alpha)$ *are periodic and complete with the possible exception of two, one of which may be periodic without being complete and the other complete without being periodic.*

Proof. In view of Corollary 1.3 it suffices to show that the above conditions imply the degeneration of the first nilpotent (or, simply, metabelian) products into direct products. Further, Theorem 5.1 in [5] allows us to restrict ourselves to the case of two factors. Finally, Theorem 5.3 in that paper allows us to assume that conditions 1) or 2) hold for the factors A_α rather than for the factor groups $A_\alpha/(A_\alpha, A_\alpha)$.

That the metabelian product of two periodic groups A and B with co-prime characteristics degenerates into a direct product follows from the fact that in case all the commutators (a, b) with $a \in A$ and $b \in B$ are 1 (see Theorem 1.6, [5]).

Now let the group A be complete and the group B periodic. Their metabelian commutator subgroup $(A, B)_Z$ is generated by all commutators (a, b), with $a \in A$, $b \in B$. For these commutators we have the equality $(a^k, b) = (a, b^k)$ (see [5], 1.5.1). Now let the element b be of order n. The completeness of the group A implies the existence in A of an element x such that $x^n = a$. Hence, $(a, b) = (x^n, b) = (x, b^n) = (x, 1) = 1$. It follows that in the case under consideration the metabelian commutator subgroup $(A, B)_Z$ consists of the unit element only, which implies that the metabelian product considered coincides with the direct product of A and B.

1.5. We note a few important instances of products satisfying conditions 1) or 2) above:

A) *All nilpotent products of p-groups for different p's degenerate into direct products.*

B) *All nilpotent products which coincide with their commutator subgroups degenerate into direct products.*

The following assertion is obvious (see, e.g., [5], 4.6):

C) *A nilpotent product of a group* A *coinciding with its commutator subgroup and an arbitrary group* B *degenerates into a direct product.*

Theorem 1.6 below again points to the many similarities between groups which coincide with their commutator subgroups and groups without center. We observe, however, that nilpotent products of groups without center need not degenerate into direct products; this in distinction to the case of nilpotent products of groups which coincide with their commutator subgroups. Indeed, the metabelian product of two symmetric groups of orders n and m is a group of order $2 \cdot n!\, m!$ (see 4.7 and 4.2 of [5]) and, consequently, differs from their direct product. (It is clear that in cases of this nature, i.e., when degeneration does not occur, the nilpotent products of groups without center have non-trivial centers.)

1.6. Theorem. *If a group G coincides with its commutator subgroup or has no center, then every one of its nilpotent decompositions degenerates into a direct decomposition.*

As regards groups which coincide with their commutator subgroups, the assertion of the theorem follows from the fact that (see [4], Ch. II, 6.8) every factor of an arbitrary decomposition of a group of this type also coincides with its commutator subgroup. The desired result now follows from criterion B) in 1.5. As for groups without center the assertion of the theorem is a consequence of the fact that in the k-th nilpotent product $\Pi_\alpha^{(k)} A_\alpha$ the minimal central series defined by the commutator subgroup $(A_\alpha)^{(k)}$ reaches 1 in at most k steps (see [4], Ch. II, 3.5), which in turn implies (see [4], Ch. I, 5.11) that $(A_\alpha)^{(k)} = 1$.

1.7. The facts presented so far show that it may be possible to decompose a group into nilpotent products with different numbers k. However, in all known examples of this type one of any two decompositions degenerates into the other. In other words, the following formulation of the question B) raised in the introduction presents an unsolved problem:

B′) Is it possible to decompose a group G into two nilpotent products

$$G = \prod_\alpha^{(k)} A_\alpha = \prod_\beta^{(l)} B_\beta ,$$

where the product $\Pi_\alpha^{(k)} A_\alpha$ does not (under the assumption that $k > l$) degenerate into the l-th nilpotent product?

§2. Isomorphic decompositions of a finite group into like-numbered (i.e., having the same number k) nilpotent products of cyclic groups.

2.1. The following result is an analogue of the theorem on the isomorphism of two decompositions of a finite abelian group into a direct product of primary cyclic subgroups. It is the simplest result bearing on the problem of the connections between two like-numbered nilpotent products of a group.

Theorem. *If (a finite) group G can be decomposed into the k-th nilpotent product of a finite number of its primary cyclic subgroups:*

$$G = A_1 (k) A_2 (k) \dots (k) A_n, \tag{1}$$

$A_i = \{a_i\}$, *then every nilpotent decomposition*

$$G = B_1 (k) B_2 (k) \dots (k) B_m \tag{2}$$

of G with factors B_j which cannot be decomposed into a k-th nilpotent product is isomorphic to the decomposition (1), i.e., $m = n$ and, furthermore, it is possible to *to set up a one-to-one correspondence between the factors* A_i *and* B_j *under which corresponding factors are isomorphic.*

The proof given below follows closely the well-known proof ([2], p. 100) of an analogous theorem for the abelian case.

Let $a_1 = b_1 b_2 \cdots b_m v$ be the regular representation (see [4], Ch. II, 1.2) of the element a_1 with respect to the decomposition (2). Then (see [4], Ch. II, 1.1 and 1.3) $b_j \in B_j$, $v \in V = \overline{(B_1, B_2, \cdots, B_m)}$. The orders of all the regular components b_j divide the order p^{α} of the element a_1. We now write down the regular representations of all the elements b_j with respect to the decomposition (1):

$$b_j = a_1^{\beta_{j1}} a_2^{\beta_{j2}} \ldots a_n^{\beta_{jn}} u_j \quad (j = 1, 2, \ldots, m), \tag{3}$$

where $u_j \in U = \overline{(A_1, A_2, \cdots, A_n)} = (G, G)$ (see [4], Ch. II, 6.8). Since $V \subseteq (G, G) = U$, the regular components (with respect to the decomposition (1)) of the element v are all equal to 1, so that $a_1 = a_1^{\beta_{11}+\beta_{21}+\cdots+\beta_{m1}}$; whence, $\beta_{11} + \beta_{21} + \cdots + \beta_{m1} \equiv 1 (p^{\alpha})$. Hence, at least one β_{i1} and p must be co-prime. For definiteness let us put:

$$(\beta_{11}, p) = 1. \tag{4}$$

Then the order of the element b_1 is exactly p^{α}.

We introduce the notations:

$$B_1' = \{b_1\},$$
$$A_1^* = A_2 (k) \ldots (k) A_n,$$
$$H = \{B_1', A_1^*\},$$

and we prove that $H = G$. It is clear that for this it suffices to show that $A_1 \subset H$.

Putting $j = 1$ in (3) we get the relation: $a_1^{-\beta_{11}} = a_2^{\beta_{12}} \ldots a_n^{\beta_{1n}} u_1 b_1^{-1}$, whence, in view of (4),

$$a_1 = b_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n} w = b_1^{\alpha_1} a_1^* w, \tag{5}$$

where $w \in (G, G)$ and $a_1^* \in A_1^*$. This representation for the element a_1 implies that we will have proved the inclusion $A_1 \subset H$ as soon as we show that $(G, G) = (H, H)$. Using the notations introduced above, we can write: $G = A_1 (k) A_1^*$ and $(G, G) = (A_1, A_1^*)(A_1^*, A_1^*)$ (by 6.8, Ch. II, [4]). In view of Hall's relations (see 2.1.2 and 2.1.3, Ch. I, [4]) we have the following expansion for a commutator (a_1, a^*), $a^* \in A_1^*$:

$$\begin{aligned}(a_1, a^*) &= (b_1^{\alpha_1} a_1^* w, a^*) = (b_1^{\alpha_1}, a^*)((b_1^{\alpha_1}, a^*), a_1^* w)(a_1^* w, a^*) = \\ &= (b_1^{\alpha_1}, a^*)((b_1^{\alpha_1}, a^*), w)(w, (a_1^*, (b_1^{\alpha_1}, a^*)))((b_1^{\alpha_1}, a^*), a_1^*) \times \\ &\quad \times (a_1^*, a^*)((a_1^*, a^*), w)(w, a^*).\end{aligned}$$

This expansion shows that the commutator (a_1, a_1^*) is contained in the subgroup $J = \{(H, H), ((G, G), H)\}$. The same conclusion holds for commutators (a_1^γ, a^*), where a_1^γ is a power of a_1. But commutators of this form generate the commutator subgroup (A_1, A_1^*). Hence, $(A_1, A_1^*) \subseteq J$. This implies that $(G, G) \subseteq J$. Consequently,

$$^1G \equiv (G, G) = \{^1H, (^1G, H)\}. \tag{6}$$

As the k-th nilpotent product of abelian groups, the group G is a nilpotent group of class not higher than k. Consequently, iterating (6) $k-1$ times, we are led to the equality $^1G = {}^1H$. Indeed, by (6), every element $^1g_\alpha$ in ^1G is of the form:

$$^1g_\alpha = {}^1_\alpha h_1 \, ({}^1_\alpha g_{11}, \; {}_\alpha h_{11}) \, ({}^1_\alpha g_{12}, {}_\alpha h_{12}) \cdots ({}^1_\alpha g_{11}, {}_\alpha h_{1s_1}) \cdot {}^1_\alpha h_2 \, ({}^1_\alpha g_{21}, {}_\alpha h_{21}) \cdots. \tag{7}$$

Replacing in this representation of $^1g_\alpha$ the elements $^1g_{11}, {}^1g_{12}, \cdots, {}^1g_{21}, \cdots$ by expressions analogous to (7) and transforming every commutator $(^1g_{11}, h_{11}), \cdots,$ by applying relations 2.1.2–3, Ch. I, [4], as many times as necessary, we find that in the transformed expression for $^1g_\alpha$ elements from ^1G occur only as inner components (see [4], Ch. I, 1.5) of n-fold commutators, $n \geq 2$. Replacing once more elements from ^1G by appropriate expressions of the form (7), we find that elements from ^1G occur only as inner components of n-fold commutators, $n \geq 3$, etc. Since, G is a nilpotent group such that all its $(k + 1)$-fold commutators are equal to 1, it follows that by replacing $k - 1$ times the components from ^1G by appropriate expressions of the form (7) in the representation of an element ^1g from ^1G, we obtain a representation of ^1g in the form of a product of elements ^1h_i and commutators whose components belong to H. In other words, ^1g is itself an element of (H, H). Thus, $(G, G) = (H, H)$.

The representation (5) of the element a_1 now shows that $a_1 \in H$, i.e., $H = G$, or,

$$G = A_1 \, (k) \, A_1{}^* = \{B_1{}', \; A_1{}^*\},$$

where $B_1' \cong A_1$. Consequently, by Theorem 6.11, Ch. II, [4], the group $G = \{B_1', A_1^*\}$ is a factor group of the product $B_1' \, (k) \, A_1^*$. However, a finite group cannot be isomorphic to a proper factor group of itself. This fact permits us to conclude that $\{B_1', A_1^*\} = B_1' \, (k) \, A_1^*$.

Thus,

$$G = B_1{}' \, (k) \, A_2 \, (k) \ldots (k) \, A_n = B_1 \, (k) \, B_2 \, (k) \ldots (k) \, B_m,$$

where $B_1' \cong A_1$ and $B_1' \subseteq B_1$. Continuing the replacement of the factors $A_2, \cdots,$ A_n in the indicated manner, we find that $G = B_1' \, (k) B_2' \, (k) \cdots (k) \, B_n'$, where

where $B_i' \cong A_i$, $B_i' \subseteq B_{j_i}$ $(i = 1, 2, \cdots, n)$. Now, the B_i' generate the whole group G so that $m \leq n$. It now follows from the assumed indecomposability of the factors B_j that $B_j' = B_{j_i}$, $m = n$, and $(j_1, j_2, \cdots, j_n)$ is a permutation of the numbers $(1, 2, \cdots, n)$.

2.2. Remark. The analog for direct products of the theorem just proved is valid for all finite abelian groups. This is due to the fact that every finite abelian group can be decomposed into a direct product of primary cyclic subgroups. In distinction to this the k-th nilpotent products of primary cyclic groups, which, by Theorem 6.11, Ch. II,[4], are groups of class not higher than k, do not, as is well-known, exhaust the class of all nilpotent groups of class not higher than k. Thus, e.g., the results of [5] imply that even the class of finite metabelian groups (i.e., nilpotent groups of class 1) is far from being exhausted by all the first nilpotent products of primary cyclic groups. The main difficulty in the study of metabelian groups consists in the fact that in addition to the first nilpotent products of cyclic groups and their regular factor groups the class of metabelian groups comprises groups which cannot be decomposed into any regular product of their subgroups. Such metabelian groups are, e.g., the non-abelian groups of order p^3 different from those considered in 4.31 and 4.32 of [5]. One such group is the group of quaternions.

We observe, however, that Theorem 6.11, Ch. II, [4] implies that the k-th nilpotent products of a finite number of primary cyclic groups form an important class of nilpotent groups whose factor groups exhaust the class of all finite nilpotent groups of class not higher than k. Theorem 2.1 permits us to refer to groups which admit of such decompositions as "regular" nilpotent groups, which, like finite abelian groups, can be prescribed by means of systems of numerical invariants.

§3. Extension of a theorem of Baer and Levi to nilpotent products.

3.1. We now show that a well-known theorem of Baer and Levi (see, e.g., [1], p. 301) concerning the impossibility of simultaneous decomposition of a group into a direct product and a free product admits of the following extension:

Theorem. *A group G which can be decomposed into a free product of its subgroups cannot be decomposed into a nilpotent product of its subgroups.*

Assume that it is possible to decompose the group G into a nilpotent product $G = \Pi_\alpha^{(k)} A_\alpha$. By 3.5, Ch. II, [4], ${}_k(A_\alpha)^{(k)} = 1$. The fact that the group G can be decomposed into a free product shows that it is a group without center. Therefore, by 5.11, Ch. I, [4], $(A_\alpha)^{(k)} = 1$, i.e., $G = \Pi_\alpha^{(k)} A_\alpha$ degenerates into a direct product. However, by the theorem of Baer and Levi the group G cannot be decomposed into a direct product.

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DESCRIPTION OF ALL IRREDUCIBLE UNITARY REPRESENTATIONS OF THE CLASSICAL GROUPS

M. A. NAĬMARK

In the joint works of the author and I. M. Gel'fand [2–5] * there were described and fully investigated the so-called principal and supplementary series of irreducible unitary representations of the complex classical groups. For the case of the complex unimodular group of the second order, in one of these works [2] it was proved that the representations in the principal and supplementary series exhaust (up to equivalence) the irreducible unitary representations of this group.

The question of whether this proposition remains valid for an arbitrary classical group up to now has remained open. It has been answered affirmatively for representations of the complex unimodular groups containing the unit representations of the unitary subgroup [5]. The proof of this result was carried out by a method essentially different from the proof in [2] and based on a study of the maximal ideals of certain commutative rings.

In this note there is given a complete solution of the above formulated question for an arbitrary complex classical group. The basic ideas of the proof presented below consist in the combination of modifications and generalizations of the methods of papers [2, 5] with certain new results of an algebraic character and with certain new norm estimates in a group ring.

Thus, the following theorem holds:

Every irreducible unitary representation of a complex classical group is equivalent to one of its representations in the principal or supplementary series.

For simplicity and brevity of exposition we limit ourselves to the case of a complex unimodular group, which we denote by $\mathfrak{G}$, although all considerations have a general character.

1. **The representation** $c(u)$. Let T_g be an irreducible unitary representation of the group $\mathfrak{G}$. We denote by $\mathfrak{U}$ the subgroup of all unitary matrices $u \in \mathfrak{G}$, and by Γ the subgroup of all diagonal matrices $\gamma \in \mathfrak{U}$. Then T_u is a unitary representation of the group $\mathfrak{U}$; because of the compactness of the latter T_u decomposes into a direct sum of irreducible representations, which are all finite-dimensional. As is well known (see, for example, [1]), every such representation is described with the help of a certain character $\chi(\gamma)$ of the group Γ, which is called the weight of the representation. It follows from this that this representation may be given with the help of integers $m_1, m_2, \cdots, m_{r-1}$ such that

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* In the present note the terminology and results of these papers are used.

$$m_1 \geq m_2 \geq \cdots \geq m_{r-1} \geq 0,$$

where r is the dimension of the representation. Therefore the weights of the irreducible representations of the group $\mathfrak{U}$ met in the decomposition of the representation T_u can be arranged in a lexicographic order. Then among these representations there will be a representation of lowest weight, which we denote by $c(u)$. We denote the weight of the representation $c(u)$ by $\chi(\gamma)$, and its dimension by r.

I. *The representation* $c(u)$ *occurs only once in the decomposition of the representation* T_u *into irreducible ones.*

2. **The ring** R_0'. We denote by R the group ring of the group $\mathfrak{G}$, and by R_0' the set of all functions $x(g) \in R$ of the form

$$x(g) = x(u_1^{-1}\,\epsilon\, u_2) = r^2 \sum_{p,q=1}^{r} X_{pq}(\epsilon)\,\overline{c_{p1}(u_1)}\,c_{q1}(u_2),$$

where ϵ is a diagonal matrix with positive diagonal elements $\epsilon_1, \epsilon_2, \cdots, \epsilon_r$ and $\epsilon \in \mathfrak{G}$, and $(c_{pq}(u))$ is the matrix of the operator $c(u)$ relative to some base. We denote the set of all matrices ϵ by E.

II. *The set* R_0' *is a subring (without unity) of the ring* R. *With proper normalization of the invariant measures* $d\mu(\epsilon)$, $d\mu(g)$

$$\frac{1}{r^2} \sum_{p,q=1}^{r} |X_{pq}(\epsilon)|\,\omega(\epsilon) \leq \int |x(g)|\,d\mu(g) \leq r^2 \sum_{p,q=1}^{r} |X_{pq}(\epsilon)|\,\omega(\epsilon)\,d\mu(\epsilon), \tag{1}$$

where

$$\omega(\epsilon) = \sum_{1 \leq p < q <} (\epsilon_p^2 - \epsilon_q^2)^2. \tag{2}$$

3. **The functions** $B(\delta)$. Set* for $x(g) \in R'$

$$B_{pq}(\delta) = \beta^{-\frac{1}{2}}(\delta) \int x(u_1^{-1}\delta\,\zeta u_2)\,c_{p1}(u_1)\,\overline{c_{q1}(u_2)}\,d\mu(u_1)\,d\mu(u_2)\,d\mu(\zeta); \tag{3}$$

the matrix with elements $B_{qp}(\delta)$ we denote by $B(\delta)$.

III. *The matrix* $B(\delta)$ *commutes with* $c(\gamma)$ *and* $B(\gamma\delta) = c(\gamma)B(\delta) = B(\delta)c(\gamma)$.

IV. *The correspondence* $x(g) \to B(\delta)$ *has the following properties:*

1) *if* $x(g) \to B(\delta)$, *then* $x^*(g) \to (B(\delta^{-1}))^*$;

2) *if* $x_1(g) \to B^{(1)}(\delta)$, $x_2(g) \to B^{(2)}(\delta)$, *then* $\lambda_1 x_1(g) + \lambda_2 x_2(g) \to \lambda_1 B^{(1)}(\delta) + \lambda_2 B^{(2)}(\delta)$ *and* $\int x_1(g_1) x_2(g g_1^{-1})\,d\mu(g_1) \to \int B^{(1)}(\delta') B^{(2)}(\delta\delta'^{-1})\,d\mu(\delta')$.

4. **The rings** R'' **and** B. We choose an orthonormal base $e_1, e_2, \cdots, e_r$ of the space of the representation $c(u)$ such that every matrix $c(\gamma)$ is diagonal; this is possible because the matrices $c(\gamma)$ commute among themselves. Then the base vectors $e_1, e_2, \cdots, e_r$ are weight vectors of the representation $c(u)$. Let

* Concerning the meaning of δ, ζ, $\beta(\delta)$ see (4), §§1–4.

e_1 be the vector for the highest weight; this weight is, by construction, the weight $\chi(\gamma)$ of the representation $c(u)$. The weight $\chi(\gamma)$ is distinct from the weights $\chi_2(\gamma), \cdots, \chi_r(\gamma)$ of all the other base vectors $e_2, \cdots, e_r$ because, as is well known, in the space of the irreducible representation $c(u)$ there is (up to a numerical factor) only one vector of highest weight. It follows from this circumstance and from III that $B_{1q}(\delta) = B_{p1}(\delta) = 0$ for $p \neq 1$, $q \neq 1$.

We denote by $\mathfrak{B}$ the set of all functions $B_{11}(\epsilon)$. From the preceding remark and from IV we conclude:

V. *The correspondence* $x(g) \to B_{11}(\epsilon)$ *has the following properties:*

1) *if* $x(g) \to B_{11}(\epsilon)$ *then* $x^*(g) \to B_{11}(\epsilon^{-1})$;

2) *if* $x_1(g) \to B_{11}^{(1)}(\epsilon)$, $x_2(g) \to B_{11}^{(2)}(\epsilon)$ *then* $\lambda_1 x_1(g) + \lambda_2 x_2(g) \to \lambda_1 B_{11}^{(1)}(\epsilon) + \lambda_2 B_{11}^{(2)}(\epsilon)$, $\int x_1(g_1) x_2(g g_1^{-1})\, d\mu(g_1) \to \int B_{11}^{(1)}(\epsilon') B_{11}^{(2)}(\epsilon \epsilon'^{-1})\, d\mu(\epsilon')$.

With operations defined in this way $\mathfrak{B}$ is a commutative ring with an involution.

5. The group S_0. We denote by S the set of all matrices s representing permutations of the vectors $e_1, e_2, \cdots, e_r$. If $\gamma \in \Gamma$ then $s^{-1}\gamma s \in \Gamma$. Furthermore, we denote by S_0 the set of all matrices $s \in S$ such that $\chi(s^{-1}\gamma s) = \chi(\gamma)$ for every $\gamma \in \Gamma$. Clearly S_0 is a subgroup of the group S.

We denote by R_0'' the set of all functions $x(g) \in R_0'$ such that $B_{pq}(\delta) = 0$ whenever $\chi_p(s^{-1}\epsilon s) \neq \chi(\epsilon)$ or $\chi_q(s^{-1}\epsilon s) \neq \chi(\epsilon)$ for all $s \in S$.

VI. R_0'' *is a commutative subring of the ring* R_0' *(and therefore of* R*) isomorphic to the ring* $\mathfrak{B}$.

VII. *The functions* $B_{11}(\epsilon) \in \mathfrak{B}$ *satisfy the condition* $B_{11}(s^{-1}\epsilon s) = B_{11}(\epsilon)$ *for every* $s \in S$.

6. The ring $\mathfrak{B}'$. We denote by $\mathfrak{B}'$ the set of functions $\phi(\epsilon)$, continuously differentiable up to the $2(r-1)(r-2)$-th order inclusive, vanishing outside a compact set (depending on the function), and satisfying the condition $\phi(s^{-1}\epsilon s) = \phi(\epsilon)$ for every $s \in S_0$.

VIII. *The set* $\mathfrak{B}'$ *is a subring of the ring* $\mathfrak{B}$. *If* $\phi \in \mathfrak{B}'$ *then the corresponding function* $x(g) \in R_0''$ *is computed by the formula*

$$x(u_1^{-1}\epsilon u_2) = c \int L\phi(\epsilon')\, \overline{c_{11}(v_1 u_1)}\, c_{11}(v_2 u_2)\, dv_1,$$

where c *is a certain constant,* $v_1\epsilon = \epsilon'\zeta v_2$, *and* L *is the differential operator defined in* [4] (§26).

7. The functional $F(x)$. We define for any $x(g) \in R$,

$$F(x) = \int x(g)(T_g^{-1} e_1, e_1)\, d\mu(g).$$

Then $F(x)$ is a positive functional on R, completely determining (up to equivalence) the representation T_g.

IX. *For every function* $x(g) \in R$ *there exists a function* $\hat{x}(g) \in \mathfrak{B}$ *such that* $F(\hat{x}) = F(x)$.

Here essential use is made of the fact that $c(u)$ is the representation of lowest weight contained in T_u.

By virtue of IX our problem is reduced to finding positive functionals $F(x)$ on the ring $\mathfrak{B}$ (to which unity should be adjoined).

8. Connection with the maximal ideals of the ring $\widetilde{\mathfrak{B}}$. We denote by $\widetilde{\mathfrak{B}}$ the ring $\mathfrak{B}$ with unity formally adjoined, so that $\mathfrak{B}$ is a maximal ideal of $\widetilde{\mathfrak{B}}$. Making use of proposition IX, it can be shown that:

X. *The representations* T_g *generate the maximal symmetric ideals of* $\widetilde{\mathfrak{B}}$ *different from* $\mathfrak{B}$. *Two such representations* T_g, T_g' *generating the same maximal ideal of* $\widetilde{\mathfrak{B}}$ *are equivalent.*

Making use of proposition VIII, it can be proved (see the detailed treatment of an analogous situation in (2)) that every maximal ideal of $\mathfrak{B}$ is determined by some function

$$\chi(\epsilon) = \epsilon_1^{i\rho_1}\epsilon_2^{i\rho_2}\cdots\epsilon_r^{i\rho_r},$$

where $\rho_1, \rho_2, \cdots, \rho_r$ are complex numbers normalized by the condition

$$\rho_1 + \rho_2 + \cdots + \rho_r = 0.$$

Two such functions $\chi_1(\epsilon)$ and $\chi_2(\epsilon)$ determine the same maximal ideal if and only if $\chi_2(\epsilon) = \chi_1(s^{-1}\epsilon s)$, where $s \in S_0$. The condition of symmetry on the maximal ideal imposes supplementary limitations on the numbers $\rho_1, \rho_2, \cdots, \rho_r$. If, for example, $S_0 = \{e\}$, then all these numbers must be real, and our maximal ideal coincides with the maximal ideal determined by certain representations in the principal series; consequently, in this case the representation T_g is equivalent to a representation in the principal series. When $S_0 \neq \{e\}$ the numbers $\rho_1, \rho_2, \cdots, \rho_r$ can also be complex. In the case of real numbers $\rho_1, \rho_2, \cdots, \rho_r$ the representation T_g is, by the preceding, equivalent to a representation in the principal series. If, however, some or all of the numbers $\rho_1, \rho_2, \cdots, \rho_r$ are complex, then the condition of symmetry and the positiveness of the corresponding functional $F(x)$ on the ring R impose limitations on these numbers from which it follows that the representation T_g is equivalent to a representation in the supplementary or principal degenerate series. This completes the proof of the fundamental theorem formulated in the beginning of the note.

We denote by $\widetilde{\Gamma}$ the set of all matrices $u \in \mathfrak{U}$ for which e_1 is a proper vector of the operator $c(u)$. Clearly $\Gamma \subset \widetilde{\Gamma}$. If $\Gamma = \widetilde{\Gamma}$ then the representation T_g is nondegenerate; in the opposite case T_g is a degenerate representation.

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UNITARY REPRESENTATIONS OF THE REAL UNIMODULAR GROUP (PRINCIPAL NONDEGENERATE SERIES)

I. M. GEL'FAND AND M. I. GRAEV

In this paper an account is given of the principal nondegenerate series of unitary representations of a group of real unimodular matrices of arbitrary order, and the irreducibility of the representations obtained is proved.

Introduction

1. In [1] there was considered the totality of irreducible unitary representations of the classical complex groups. These representations were given there as representations in spaces of functions on various manifolds, on which the group acts.

The investigation of the representations of real semi-simple groups exhibits, in comparison with complex groups, a number of new peculiarities. This can be clearly seen even when we compare the representations of the group of complex unimodular second-order matrices, considered in [2], with the representations of the group of real unimodular second-order matrices, considered in [5]. We illustrate the difference in question by the principal series* of representations of these groups.

A representation of a principal series of the group of complex second-order matrices is given as follows. We consider the space of functions $f(z)$ of the complex variable $z = x + iy$ with integrable square:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(z)|^2\,dx\,dy < +\infty.$$

A representation of a principal series is given by an integer m and a number ρ, and consists in that to every matrix $g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ there corresponds a mapping, which takes an arbitrary function $f(z)$ into the function

$$T_g f(z) = f\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) |\beta z + \delta|^{m+\rho-2} (\beta z + \delta)^{-m}.$$

The group of real matrices has two principal series of representations. In the first of them we consider functions $f(x)$ of integrable square on the real axis:

$$\int_{-\infty}^{+\infty} |f(x)|^2\,dx < +\infty.$$

A representation is given by a real number ρ and consists in that to every real matrix $g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ there corresponds a mapping

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* Translator's note: Plural, unless the context indicates otherwise.

$$T_g f(x) = f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) | \beta x + \delta |^{i\rho - 1}$$

or a mapping

$$T_g f(x) = f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) | \beta x + \delta |^{i\rho - 1} \cdot \operatorname{sign} (\beta x + \delta).$$

This series of representations forms the most natural analogue of the above series of representations of the complex group.

There is, however, one further principal series of representations of the real group. A representation of this series is given by a positive integer m. It is realised in the space of functions $f(z)$ of the complex variable $z = x + iy$, which are analytic in the half-plane $y > 0$, and for which there converges the integral

$$\int_{-\infty}^{+\infty} \int_{0}^{\infty} | f(z) |^2 y^{m-2} \, dy \, dx.$$

The representation consists in that to every real matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ there corresponds the mapping

$$T_g f(z) = f\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) (\beta z + \delta)^{-m}.$$

The other part of this series may be constructed in the space of functions which are analytic in the half-plane $y < 0$. *

It is natural to pose the question of the representations of real simple groups. In particular, there arises the question of the reason for the appearance of series given in terms of analytic functions, the reason for the emergence of several series instead of one, and so on.

In this work we consider representations of a real unimodular group of arbitrary order. Here there appear already all the characteristic features of the representations of real groups. In the present paper we consider only principal series of representations of a real unimodular group. Other series of representations of this group, and again the characters of the representations, the Plancherel theorem

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* We remark that this series may also be given as a representation on the real axis, since a function $f(z)$ analytic in a half-plane is completely determined by its boundary values $f(x)$. Here the scalar product will no longer be given by the formula

$$\int_{-\infty}^{+\infty} \int_{0}^{\infty} f_1(z) \overline{f_2(z)} \, y^{m-2} \, dy \, dx,$$

but by the formula

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} | x_1 - x_2 |^{m-2} f_1(x_1) \overline{f_2(x_2)} \, dx_1 \, dx_2.$$

However, one should here consider not all functions, but only boundary functions to analytic functions from the upper half-plane, or the lower as appropriate.

and other questions will be considered later*.

2. We proceed to explain how the representations of the principal nondegenerate series of a real unimodular group are in fact constructed.

We first recall that the representations of the group $\mathfrak{G}$ of complex unimodular n-th order matrices were constructed in [1] in a space of functions $f(z)$, defined on elements z of a manifold of right cosets** of the group $\mathfrak{G}$ with respect to the subgroup K of complex triangular matrices. Of course the subgroup K and any subgroup $g_0^{-1}Kg_0$ conjugate to it lead to one and the same manifold, and consequently to precisely the same representations.

Let us pass now to the group G of real unimodular matrices. Here we must make a distinction between the subgroup K of triangular matrices and the subgroups $K^{g_0} = g_0^{-1}Kg_0$ conjugate to it of the complex group $\mathfrak{G}$, since the intersections of these subgroups with the real group G may be different. Here the subgroups K^{g_0}, which have essentially distinct intersections with G***, may lead to distinct series of representations of the group G of real matrices.

The representations which correspond to a given subgroup $K^{g_0} = g_0^{-1}Kg_0$ are constructed in a space of functions defined on elements of the manifold of right cosets of the real group G with respect to its intersection $G \cap K^{g_0}$ with the subgroup K^{g_0}. However, this space consists, generally speaking, not of all functions, but only of functions analytic in certain of the variables.

The appearance here of analytic functions is extremely interesting; it can be explained on the following ground.

Let first $\mathfrak{G}$ be the complex group and K its subgroup. The space of functions which are constant over the right cosets of the group $\mathfrak{G}$ with respect to the subgroup K may also be characterised by the condition that these functions should be constant under infinitely small displacements which correspond to multiplication on the left of elements of the group $\mathfrak{G}$ by infinitesimal elements from K. Since infinitely small displacements are given by Lie operators, it follows that these functions satisfy equations $X_i f = 0$, where X_i are Lie operators corresponding to infinitesimal elements from K.

Let us now pass to a real group. We shall consider functions $f(g)$ on a group G of real matrices. Let K_B be the subgroup of real**** triangular matrices, and $K^{g_0} = g_0^{-1}K_Bg_0$ some subgroup of the group $\mathfrak{G}$ of complex matrices which is conjugate to K_B. We first assume that the matrix g_0, by means of which the subgroup

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* The construction of representations for the general case of a semi-simple real Lie group was outlined in the authors' work [3].

** Translator's note: In this translation, as in the original, the cosets are referred to as "of the group with respect to the subgroup", rather than "of the subgroup".

*** That is to say their intersections are not conjugate subgroups in G.

**** Translator's note: The suffix "B" probably comes from the Russian word for "real".

K^{g_0} is defined, is real. Then K^{g_0} is a subgroup of the real group G. We demand, as in the case of the complex group, that the functions $f(g)$ should be constant under the infinitely small displacements which correspond to multiplication on the left of elements of the group G by infinitesimal elements from K^{g_0}. This requirement may be expressed, as in the case of the complex group, in the form of a certain system of linear differential equations $X_i f = 0$, which the functions $f(g)$ must satisfy. The coefficients of these equations will be expressible analytically in terms of the elements of the matrix g_0. We can therefore write down formally the Lie operators X_i also for the case when the matrix g_0 is no longer real, and require that the functions $f(g)$ should satisfy the corresponding system of differential equations $X_i f = 0$.

In the case when the matrix g_0 is real the system of differential equations obtained is hyperbolic, as in the case of the complex unimodular group. Its characteristics will be precisely the right cosets of the real group G with respect to the subgroup K^{g_0}. The equations $X_i f = 0$ will therefore mean that the functions $f(g)$ are constant on the cosets of G with respect to K^{g_0}. If, however, the matrix g_0 is complex the corresponding system of differential equations for the functions $f(g)$ will no longer be hyperbolic*. In this case its solution will turn out to be functions constant on the right cosets of the group G with respect to its intersection $G \cap g_0^{-1} K g_0$ with the subgroup $g_0^{-1} K g_0$, K being the group of complex triangular matrices, and in addition analytic in certain of the parameters of the group G. Thus the constancy with respect to certain parameters of the group is replaced, under the transition from a complex group to a real group, by the requirement of analyticity.

Representations of the group G of real matrices are achieved, in view of the above, in spaces of functions defined on manifolds of cosets of the group G with respect to its intersection with any of the subgroups $g_0^{-1} K g_0$, conjugate to the group K of complex triangular matrices. The totality of manifolds to be obtained in this way may also be characterised in another way. We first recall the manner in which this was done in [1] for the case of the complex unimodular group $\mathfrak{G}$. In this work there was introduced the subgroup Z of triangular matrices which have zeros above the principal diagonal, and units along the principal diagonal, i.e., the group of matrices $z \| z_{pq} \|$ such that $z_{pp} = 1$ and $z_{pq} = 0$ for $p < q$ $(p, q = 1, \cdots, n)$. It was shown that almost every matrix g of the complex group $\mathfrak{G}$ can be uniquely represented in the form $g = kz$, where k is a triangular matrix from K and z a matrix from Z. In view of this the manifold of right cosets of the complex group $\mathfrak{G}$

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* In fact the characteristics of the system in this case would have to be the cosets of the group G with respect to its intersection with the subgroup $g_0^{-1} K g_0$. Since this intersection is partly imaginary, some of the characteristics are imaginary. We remark that the analyticity of the functions in certain of the parameters (see below) in fact takes the place of constancy on the cosets with respect to an "imaginary subgroup" of the group G.

with respect to its subgroup K may be identified (to within a manifold of lower dimension) with the manifold of elements of the group Z. The mapping of the manifold Z by a matrix g of the complex group $\mathfrak{G}$ then reduces to this, that an element z of this manifold goes into an element z_1 such that $zg = kz_1$, where k is a matrix of the subgroup K.

The manifold Z is not transitive with respect to mappings by elements of the group G of real matrices, and so must fall into a certain number of transitive manifolds. Let us consider any one of these transitive manifolds in Z. Let z be any point of this manifold. Then a stationary subgroup for z will be the intersection $G \cap z^{-1}Kz$ of the group G with the subgroup $z^{-1}Kz$. Hence the manifold in question may be identified with the manifold of cosets of the group G with respect to the subgroup $G \cap z^{-1}Kz$. Conversely, almost all elements of the complex group $\mathfrak{G}$ can be represented in the form $g = kz$, where $k \in K$, $z \in Z$, and therefore almost every subgroup, conjugate to the subgroup K of triangular matrices, has the form $z^{-1}Kz$. Consequently every manifold of cosets of the group G of real matrices with respect to its intersection with a subgroup conjugate to the group K can be identified with a certain manifold in Z which is transitive with respect to mappings by elements of the group G. Thus representations of the group G of real matrices may be realised in spaces of functions defined on one or other manifold in Z, transitive with respect to mappings by elements of the group G.

In the present work we utilise only those transitive manifolds in Z which are needed to obtain the principal nondegenerate series of representations of the real group G.

3. We proceed to describe the representations which we find in this paper for the group G of real unimodular matrices.

We commence with a description of those manifolds in Z which are transitive with respect to the real group G and which lead to the required series.

Let us consider the set of matrices z in Z which can be represented in the form of the product $z = \dot{z}x$ of two matrices $\dot{z} = \| \dot{z}_{pq} \|$ and $x = \| x_{pq} \|$ from Z of the following form. The elements $\dot{z}_{12}, \dot{z}_{34}, \cdots, \dot{z}_{2m-1,\, 2m}$ of the matrix $\dot{z}$ are complex numbers whose imaginary parts are different from zero (m being a fixed number); the remaining elements of this matrix lying under the principal diagonal vanish. The matrix x is a real matrix for which $x_{12} = x_{34} = \cdots = x_{2m-1,\, 2m} = 0$. It turns out that if $2m$ is less than the order n of the matrices in question, then the set of elements z just described forms a manifold Z_m which is transitive with respect to mappings by elements of the real group G. For $2m = n$ this set falls into two transitive manifolds Z_m^+ and Z_m^-. The first of them consists of matrices $z = \dot{z}x$ for which the product $\Pi_{p=1}^{m} \operatorname{Im} \dot{z}_{2p-1,\, 2p}$ of the imaginary parts of the elements $\dot{z}_{2p-1,\, 2p}$ of the matrix $\dot{z}$ is greater than zero; the second on the other hand consists of the matrices for which this product is less than zero.

As parameters determining the element $z = \dot{z}x$ of the manifold Z_m we shall choose the complex elements $z_1 = \dot{z}_{12}, z_2 = \dot{z}_{34}, \cdots, z_m = \dot{z}_{2m-1,2m}$ of the matrix $\dot{z}$ and the real elements x_{pq} of the matrix x.

For $m < \frac{n}{2}$ the representation is realised in the space of functions $f(z) = f(z_1, \cdots, z_m, x)$, defined on Z_m, of the m complex variables $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2, \cdots, z_m = x_m + iy_m$ and the $\frac{n(n-1)}{2} - m$ real variables x_{pq}. We require that these functions should be analytic in each complex variable z_p in the upper and lower half-planes separately. (We do not, however, assume that these functions can be continued across the real axis.) We further require that for these functions and for fixed positive integers $n_1, n_2, \cdots, n_m$ there should converge the integral

$$\int |f(z)|^2 \prod_{p=1}^{m} |y_p|^{n_p-2} \prod_{p=1}^{m} dx_p\, dy_p \cdot d\mu(x).$$

($d\mu(x)$ denotes the product of the differentials dx_{pq} of the elements of the matrix x).

The representation consists in that to every real n-th order matrix g there corresponds a mapping which takes the function $f(z) = f(z_1, \cdots, z_m, x)$ into the function

$$T_g f(z) = f(z\overline{g})\, a(z, g). \tag{1}$$

Here $z\overline{g}$ denotes the transform of z by means of the matrix g. The function $a(z, g)$ which characterises the representation is defined by fixing m positive integers $n_1, n_2, \cdots, n_m$, $m + r - 1$ real numbers $\rho_1, \rho_2, \cdots, \rho_{m+r-1}$ $(r = n - 2m)$, and by fixing $r - 1$ indices $\epsilon_1, \epsilon_2, \cdots, \epsilon_{r-1}$ which take the values 0 or 1.

We proceed to specify the function $a(z, g)$, corresponding to the given numbers n_p, ρ_q and ϵ_r $(p = 1, \cdots, m; q = 1, \cdots, m + r - 1; r = 1, \cdots, r - 1)$. For this purpose let us consider a real matrix x, appearing in the decomposition $z = \dot{z}x$ of the matrix z. It can be shown that the matrix xg can be uniquely represented in the form $xg = kx_1$. Here x_1 is a matrix of the same form as the matrix x, and the matrix k has the following form:

$$k = \left(\begin{array}{ccccccccc}
\alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots 0 \\
\gamma_1 & \delta_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots 0 \\
k_{31} & k_{32} & \alpha_2 & \beta_2 & \cdots & 0 & 0 & 0 & \cdots 0 \\
k_{41} & k_{42} & \gamma_2 & \delta_2 & \cdots & 0 & 0 & 0 & \cdots 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
k_{2m-1,1} & k_{2m-1,2} & k_{2m-1,3} & k_{2m-1,4} & \cdots & \alpha_m & \beta_m & 0 & \cdots 0 \\
k_{2m,1} & k_{2m,2} & k_{2m,3} & k_{2m,4} & \cdots & \gamma_m & \delta_m & 0 & \cdots 0 \\
k_{2m+1,1} & k_{2m+1,2} & k_{2m+1,3} & k_{2m+1,4} & \cdots & k_{2m+1,2m-1} & k_{2m+1,2m} & \lambda_1 & \cdots 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
k_{n1} & k_{n2} & k_{n3} & k_{n4} & \cdots & k_{n,2m-1} & k_{n,2m} & k_{n,2m+1} & \cdots \lambda_r
\end{array}\right)$$

We introduce the notations:

$$\Lambda_p = \begin{cases} \begin{vmatrix} \alpha_p & \beta_p \\ \gamma_p & \delta_p \end{vmatrix} & \text{for } p \le m \\ \lambda_{p-m} & \text{for } p = m+1,\ m+2,\ \cdots,\ m+r \end{cases}$$

$(r = n-2m)$. It then turns out that the function $a(z, g)$ can be represented by the following formula:

$$\alpha(z, g) = \prod_{p=1}^{m} \left(\frac{\beta_p z_p + \delta_p}{\sqrt{|\alpha_p \delta_p - \beta_p \gamma_p|}} \right)^{-n_p} \prod_{p=1}^{m+\tau-1} |\Lambda_p|^{i\rho_p} \cdot \prod_{p=m+1}^{m+\tau-1} (\operatorname{sign} \Lambda_p)^{\epsilon_p} \cdot$$

$$\cdot \left\{ |\Lambda_2|^{r_1+r_2} |\Lambda_3|^{r_1+2r_2+r_3} \ldots |\Lambda_{m+\tau}|^{r_1+2r_2+\ldots+2r_{m+\tau-1}+r_{m+\tau}} \right\}^{-1/2}. \qquad (2)$$

(Here we have taken $r_1 = \cdots = r_m = 2$, $r_{m+1} = \cdots r_{m+r} = 1$.)

For $m < \frac{n}{2}$ the representation is thus realised in the function-space described above. It is determined by m positive integers $n_1, n_2, \cdots, n_m$, $m + r - 1$ real numbers $\rho_1, \rho_2, \cdots, \rho_{m+r-1}$, and again $r-1$ indices $\epsilon_1, \epsilon_2, \cdots, \epsilon_{r-1}$ taking the value 0 or 1. The representation operator T_g is given by the formulae (1) and (2).

For $m = \frac{n}{2}$ the representation is realised in a space of functions of the form described, defined not on the entire manifold Z_m, but only on one of its parts, Z_m^+ or Z_m^-. The representation is determined by m positive integers $n_1, n_2, \cdots, n_m$ and $m-1$ real numbers $\rho_1, \rho_2, \cdots, \rho_{m-1}$. The representation operator is defined in accordance with the formulae (1) and (2).

Putting $m = 0, 1, \cdots, [\frac{n}{2}]$, we obtain $[\frac{n}{2}] + 1$ different principal nondegenerate series of representations of the group G*.

4. It should not be thought that the series which have been described here exhaust the totality of series of unitary representations of the group G of real matrices. An account of all remaining series will be given later. We indicate here, purely as an example, one more series of unitary representations which exists for groups of even-order matrices. The representations of this series are remarkable in that they are defined in a space of purely analytic functions (i.e., functions which are analytic in all the parameters).

For simplicity we shall consider the group G of real fourth-order matrices.

It turns out that the manifold Z of complex matrices, on which the group G

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* We remark that the number m is equal to the number of pairs of complex conjugate eigen-values of matrices of the stationary subgroup for a fixed element of the manifold Z_m.

The question of the equivalence of these representations will, as usual, follow automatically from formulae for the characters, which will be adduced later. These formulae for the characters shed light also on the existence of distinct series. Roughly speaking, the existence of $[\frac{n}{2}] + 1$ distinct series is connected with the fact that in the group of real matrices there exists $[\frac{n}{2}] + 1$ distinct types of classes of conjugate elements. Here the characters of a representation of a series, determined by the number m, vanishes identically on matrices which have more than m pairs of complex eigen-values.

acts, falls into two manifolds Z^+ and Z^- which are transitive with respect to G and which have the same dimensionality as Z itself, together with a manifold of lower dimensionality which borders on them. Our representations are realised in fact in the space of functions, defined on one of the manifolds Z^+ and Z^-.

With a suitable choice of parameters the manifolds Z^+ and Z^- may be regarded as manifolds of pairs (z', z'') of points of the three-dimensional space $\mathfrak{B}$, $z'(z_1', z_2', z_3')$ and $z''(z_1'', z_2'', z_3'')$, whose coordinates satisfy the inequality

$$\Delta(z', z'') = \begin{vmatrix} z_1' & z_2' & z_3' & 1 \\ z_1'' & z_2'' & z_3'' & 1 \\ \bar{z}_1' & \bar{z}_2' & \bar{z}_3' & 1 \\ \bar{z}_1'' & \bar{z}_2'' & \bar{z}_3'' & 1 \end{vmatrix} > 0$$

or the inequality $\Delta(z', z'') < 0$, respectively.

The action of the group G in the three-dimensional space $\mathfrak{B}$ is in effect that a matrix $g = \| g_{pq} \|$ transforms the point $z(z_1, z_2, z_3)$ of this space into a point $z\bar{g}$ with the coordinates

$$\hat{z}_i = \frac{z_1 g_{1i} + z_2 g_{2i} + z_3 g_{3i} + g_{4i}}{z_1 g_{14} + z_2 g_{24} + z_3 g_{34} + g_{44}} \quad (i = 1, 2, 3).$$

A representation of the series under consideration is constructed in the space of functions $f(z', z'')$ of the six complex variables z_i' and $z_j''(i, j = 1, 2, 3)$, which are defined over one of the above manifolds. We require that these functions should be analytic in all the variables z_i' and z_j'' on the manifold in question. We require furthermore that for these functions and for fixed positive integral m there should converge the integral

$$\int |f(z', z'')|^2 \cdot |\Delta(z', z'')|^{m-4} dz_1' \, dz_2' \, dz_3' \, dz_1'' \, dz_2'' \, dz_3''.$$

(Here dz denotes the product $dx\,dy$, if $z = x + iy$.)

The representation is given by a positive integer m and consists in that to every fourth-order matrix $g = \| g_{pq} \|$ there corresponds a mapping which takes the function $f(z', z'')$ into the function

$$T_g f(z', z'') =$$
$$= f(z'\bar{g}, z''\bar{g}) \left[(z_1' g_{14} + z_2' g_{24} + z_3' g_{34} + g_{44}) \cdot (z_1'' g_{14} + z_2'' g_{24} + z_3'' g_{34} + g_{44}) \right]^{-m}.$$

The series we have described has interest in connection with automorphic functions of several variables.

5. We pass to a brief account of the contents of the individual sections of the present article.

§§1-3 are of a preparatory character. In these sections are studied certain of the subgroups of the group $\mathfrak{G}$ of complex matrices and of the group G of real matrices. A study of these subgroups is essential to an account of the representations of the principal nondegenerate series of the group G.

§4 contains an account of certain manifolds in the subgroup Z of complex triangular matrices which are transitive with respect to the group G. With the help of these manifolds are subsequently constructed the representations of the principal series of the real group G.

In §5 we prove that the function-spaces on which the representations of the principal series are realised consist of functions which are analytic in certain of the parameters of the group G.

Finally in §6 is given the construction of all principal nondegenerate series of unitary representations of the real group G.

§7 is devoted to the proof of the irreducibility of the representations obtained. The proof is based upon the same idea as for the case of a group of a complex matrices (see [1]), but its execution is bound up with certain extra difficulties which are peculiar to the case of a group of real matrices.

§1. Certain subgroups of the groups $\mathfrak{G}$ and G

An analytic account of the representations of the principal series is bound up with the consideration of certain subgroups of the complex unimodular group $\mathfrak{G}$ and the real unimodular group G.

In what follows we shall express all matrices as matrices of boxes. We fix an integer $m \le \frac{n}{2}$ and put $r_1 = \cdots = r_m = 2$, $r_{m+1} = \cdots = r_{m+r} = 1$ $(r = n - 2m)$. We shall write matrices g from $\mathfrak{G}$ or from G in the form

$$g = \| g_{pq} \| \quad (p, q = 1, \cdots, m + r), \tag{1.1}$$

where g_{pq} denotes a matrix which consists of r_p rows and r_q columns. In connection with this notation we consider certain subgroups.

1. The subgroups $\hat{K}'_{m,r}$ and $\hat{K}_{m,r}$. The subgroup $\hat{K}'_{m,r}$ consists of matrices $k' = \| k'_{pq} \|$ of the complex group $\mathfrak{G}$ satisfying the condition

$$k'_{pq} = 0 \quad \text{for} \quad p > q. \tag{1.2}$$

The subgroup $\hat{K}_{m,r}$ consists of *real* matrices $k = \| k_{pq} \|$ which satisfy the condition (1.2).

2. The subgroups $Z'_{m,r}$ and $Z_{m,r}$. The subgroup $Z'_{m,r}$ consists of matrices $\zeta' = \| \zeta'_{pq} \|$ of the complex group $\mathfrak{G}$ satisfying the conditions

$$\zeta'_{pq} = 0 \quad \text{for} \quad p > q \quad \text{and the} \quad \zeta'_{pp} \quad \text{are unit matrices.} \tag{1.3}$$

The subgroup $Z_{m,r}$ consists of *real* matrices $\zeta = \| \zeta_{pq} \|$ which satisfy the condition (1.3).

3. The subgroups $D'_{m,\tau}$ and $D_{m,\tau}$. The subgroup $D'_{m,\tau}$ consists of matrices $d' = \| d'_{pq} \|$ of the complex group $\mathfrak{G}$ which satisfy the condition

$$d'_{pq} = 0 \quad \text{for} \quad p \neq q. \tag{1.4}$$

The subgroup $D_{m,\tau}$ consists of *real* matrices $d = \| d_{pq} \|$ which satisfy the condition (1.4).

The matrices d of the subgroup $D'_{m,\tau}$, and similarly for the subgroup $D_{m,\tau}$, will for brevity be written in row form

$$d = [d_1, d_2, \cdots, d_{m+\tau}], \tag{1.5}$$

where $d_p = d_{pp}$ $(p = 1, \cdots, m+\tau)$.

4. The subgroups $\hat{D}_{m,\tau}$ and $\hat{D}_{m,\tau}$. The subgroup $\hat{D}'_{m,\tau}$ consists of matrices

$$\hat{d}' = [\hat{d}'_1, \hat{d}'_2, \cdots, \hat{d}'_{m+\tau}]$$

of the group $\mathfrak{G}$ which satisfy the condition

$$|\det \hat{d}'_p| = 1 \quad (p = 1, 2, \ldots, m+\tau). \tag{1.6}$$

The subgroup $\hat{D}_{m,\tau}$ consists of *real* matrices satisfying the condition (1.6).

5. The subgroup $\dot{D}_{m,\tau}$. The subgroup $\dot{D}_{m,\tau}$ consists of real matrices

$$\dot{d} = [\dot{d}_1, \dot{d}_2, , \ldots, \dot{d}_{m+\tau}],$$

satisfying the conditions:

$$d_{m+1} = \ldots = d_{m+\tau-1} = 1, \quad |d_{m+\tau}| = 1,$$

while the matrices $\dot{d}_p$ $(p \leq m)$ have the form

$$\dot{d}_p = \begin{pmatrix} h_{11}^{(p)} & 0 \\ h_{21}^{(p)} & h_{22}^{(p)} \end{pmatrix}, \text{ with } |\det \dot{d}_p| = 1 \text{ and } h_{22}^{(p)} > 0. \tag{1.7}$$

It is easily seen that a right-invariant measure in the subgroup $\dot{D}_{m,\tau}$ is given by the formula

$$d\mu_r(\dot{d}) = d\mu_r(\dot{d}_1)\, d\mu_r(\dot{d}_2) \ldots d\mu_r(\dot{d}_m), \tag{1.8}$$

where $d\mu_r(\dot{d}_p)$ $(p = 1, \cdots, m)$ are right-invariant measures in the group of the second-order matrices $\dot{d}_p$. These measures are in turn expressible by the formulae

$$d\mu_r(\dot{d}_p) = dh_{21}^{(p)}\, dh_{22}^{(p)} \tag{1.9}$$

(see [2]).

We introduce the element

$$\dot{z}_0 = [v_1^0, v_2^0, \ldots, v_m^0, 1, \ldots, 1] \tag{1.10}$$

of the subgroup of triangular matrices Z, where

$$v_1^0 = v_2^0 = \ldots = v_m^0 = v^0 = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}. \tag{1.11}$$

Let the element $\dot{z}_0$ be transformed by means of the element $\dot{d}$ of the subgroup $\dot{D}_{m,r}$ into

$$\dot{z} = \dot{z}_0 \bar{\dot{d}}. \tag{1.12}$$

It is obvious that

$$\dot{z} = [v_1, v_2, \cdots, v_m, 1, \cdots, 1], \tag{1.13}$$

where

$$v_p = \begin{pmatrix} 1 & 0 \\ z_p & 1 \end{pmatrix}, \quad z_p = x_p + iy_p \quad (p = 1, \ldots, m). \tag{1.14}$$

It follows from the equation (1.12) that

$$v_0 \bar{\dot{d}}_p = v_p,$$

whence

$$z_p = \frac{h_{11}^{(p)} i + h_{21}^{(p)}}{h_{22}^{(p)}} = \pm \frac{1}{(h_{22}^{(p)})^2} i + \frac{h_{21}^{(p)}}{h_{22}^{(p)}},$$

that is to say

$$x_p = \frac{h_{21}^{(p)}}{h_{22}^{(p)}}, \quad y_p = \pm \frac{1}{(h_{22}^{(p)})^2}.$$

Hence

$$dx_p = \frac{1}{h_{22}^{(p)}} dh_{21}^{(p)} - \frac{h_{21}^{(p)}}{(h_{22}^{(p)})^2} dh_{22}^{(p)},$$

$$dy_p = \mp \frac{2}{(h_{22}^{(p)})^3} dh_{22}^{(p)}.$$

Consequently

$$dx_p dy_p = 2 \frac{1}{(h_{22}^{(p)})^4} dh_{21}^{(p)} dh_{22}^{(p)} = 2 \,|\operatorname{Im} z_p|^2 dh_{21}^{(p)} dh_{22}^{(p)}.$$

In virtue of (1.9) we obtain

$$d\mu_r(d_p) = \frac{1}{2(\operatorname{Im} z_p)^2} dx_p dy_p. \tag{1.15}$$

Thus the right-invariant measure in the group $\dot{D}_{m,r}$ is determined in terms of the parameters x_p, y_p $(p = 1, 2, \cdots, m)$ by the formula

$$d\mu_r(\dot{d}) = \frac{1}{2^m} \prod_{p=1}^{m} (\operatorname{Im} z_p)^{-2} \cdot dx_1 dy_1 \ldots dx_m dy_m. \tag{1.16}$$

6. **The subgroup $U_{m,r}$.** The subgroup $U_{m,r}$ consists of real matrices

$$u = [u_1, u_2, \cdots, u_m, 1, \cdots, 1],$$

where u_p is the orthogonal matrix:

$$u_p = \begin{pmatrix} u_1^{(p)} & -u_2^{(p)} \\ u_2^{(p)} & u_1^{(p)} \end{pmatrix}, \quad (u_1^{(p)})^2 + (u_2^{(p)})^2 = 1. \tag{1.17}$$

7. **The subgroup $C_{m,r}$.** The subgroup $C_{m,r}$ consists of real matrices

$$c = [c_1, c_2, \cdots, c_{m+r}],$$

satisfying the condition: if $p \leq m$ then c_p is a diagonal matrix of the form

$$c_p = \begin{pmatrix} \sqrt{\lambda_p} & 0 \\ 0 & \sqrt{\lambda_p} \end{pmatrix}, \quad \lambda_p > 0. \tag{1.18}$$

8. **The subgroups $X_{m,r}$ and $\hat{Z}_{m,r}$.** The subgroup $X_{m,r}$ consists of real matrices $x = \| x_{pq} \|$ which satisfy the condition:

$$x_{pq} = 0 \quad \text{for} \quad p < q \quad \text{while the} \quad x_{pp} \quad \text{are unit matrices.} \tag{1.19}$$

It is easy to see that the group $X_{m,r}$ possesses the twosided-invariant measure

$$d\mu(x) = \prod_{p>q} d\mu(x_{pq}), \tag{1.20}$$

where $d\mu(x_{pq})$ is the product of the differentials of all elements of the matrix x_{pq} (cf. [1], formula (11.18)].

The subgroup $\hat{Z}_{m,r}$ consists of complex matrices $\hat{z} = \| z_{pq} \|$ satisfying the condition (1.19).

In what follows we shall as a rule omit the index r in the notation for the subgroups.

§2. Canonical decompositions

1. **Decomposition of elements of the groups $\mathfrak{G}$ and G.** According to [1], almost every matrix g of the complex group $\mathfrak{G}$ can be represented, and moreover uniquely, in the form

$$g = \hat{k}'\hat{z}, \tag{2.1}$$

where $\hat{k}' \in \hat{K}'_m$, $\hat{z} \in \hat{Z}_m$. An exception is formed only by matrices for which at any rate one of the minors $g_{r_1+r_2+\cdots+r_p+1}$ vanishes.

We recall that g_p denotes a minor of the form

$$g_p = \begin{vmatrix} g_{pp} \cdots g_{pn} \\ \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ g_{np} \cdots g_{nn} \end{vmatrix}.$$

The same holds likewise for real matrices $g \in G$. In fact almost every real matrix of the group G can be represented, and uniquely, in the form

$$g = \hat{k}x, \tag{2.1'}$$

where $\hat{k} \in \hat{K}_m$, $x \in X_m$.

Also obtained in [1] were analytic expressions for the elements of the matrices k and x in the formula (2.1) or (2.1') in terms of the elements of the matrix g, which we shall now produce.

We shall denote by $\begin{pmatrix} p_1 p_2 \cdots p_m \\ q_1 q_2 \cdots q_m \end{pmatrix}$ the minor of the matrix g formed of the elements with the row-numbers $p_1, p_2, \cdots, p_m$ and the column-numbers $q_1, q_2, \cdots, q_m$

and by g_m, as above, the minor

$$\begin{pmatrix} m & m+1 & \dots & n \\ m & m+1 & \dots & n \end{pmatrix}.$$

Let $\hat{k} = \| k_{pq} \|$ and $x = \| x_{pq} \|$. Then the elements $k_{pq}^{(\lambda, \mu)}$ of the matrix k_{pq} can be found from the formulae

$$k_{pq}^{(\lambda, \mu)} = \frac{1}{g_{r_1 + \dots + r_q + 1}} \begin{pmatrix} r_1 + \dots + r_{p-1} + \lambda, & r_1 + \dots + r_q + 1, & r_1 + \dots + r_q + 2, & \dots, & n \\ r_1 + \dots + r_{q-1} + \mu, & r_1 + \dots + r_q + 1, & r_1 + \dots + r_q + 2, & \dots, & n \end{pmatrix} \tag{2.2}$$

The determinant Λ_p of the matrix k_{pp} is given by the equation

$$\Lambda_p = \frac{g_{r_1 + \dots + r_{p-1} + 1}}{g_{r_1 + \dots + r_p + 1}}, \tag{2.3}$$

and the matrices x_{pq} are found from the expressions

$$x_{pq} = h_{pp}^{-1} h_{pq}, \tag{2.4}$$

where the elements $h_{pq}^{(\lambda, \mu)}$ of the matrices h_{pq} are determined by the formulae

$$h_{pq}^{(\lambda, \mu)} = \frac{1}{g_{r_1 + \dots + r_p + 1}} \begin{pmatrix} r_1 + \dots + r_{p-1} + \lambda, & r_1 + \dots + r_p + 1, & r_1 + \dots + r_p + 2, & \dots, & n \\ r_1 + \dots + r_{q-1} + \mu, & r_1 + \dots + r_p + 1, & r_1 + \dots + r_p + 2, & \dots, & n \end{pmatrix}. \tag{2.5}$$

(cf. [1], formulae (12.16), (12.13), (12.14) and (12.15)).

2. Decomposition of the elements of the groups $\hat{K}'_m$ and $\hat{K}_m$. Any element $k' = \| k'_{pq} \|$ of the group $\hat{K}'_m$ can be represented, and moreover uniquely, in the form

$$k' = \zeta' d', \tag{2.6}$$

where $\zeta' \in Z'_m$, $d' \in D'_m$. The element of the matrix d',

$$d' = [d'_1, d'_2, \dots, d'_{m+r}],$$

are determined from the formulae

$$d'_p = k'_{pp} \quad (p = 1, \dots, m + r). \tag{2.7}$$

In just the same way an arbitrary element k of the real group $\hat{K}_m$ can be represented, and uniquely, in the form

$$k = \zeta d, \tag{2.8}$$

where $\zeta \in Z_m$, $d \in D_m$.

An arbitrary element d' of the group D'_m can in turn be represented in the form

$$d' = c\hat{d}', \tag{2.9}$$

where $c \in C_m$, $\hat{d} \in \hat{D}'_m$. This representation will be the only one, if we require in addition that the matrix

$$c = [c_1, c_2, \dots, c_{m+r}]$$

should satisfy the condition:

$$c_p > 0 \quad \text{for} \quad p > m. \tag{2.10}$$

Similarly, any element d of the real group D_m can be represented in the form

$$d = c\hat{d}, \tag{2.11}$$

where $c \in C_m$, $\hat{d} \in \hat{D}_m$. It is easy to verify that such a representation is unique if we impose the additional condition that the matrix

$$\hat{d} = [\hat{d}_1, \hat{d}_2, \cdots, \hat{d}_{m+r}]$$

should satisfy the condition:

$$\hat{d}_{m+1} = \cdots = d_{m+r-1} = 1. \tag{2.12}$$

In accordance with [2], an arbitrary real second-order matrix g_2 with determinant ± 1 can be represented in the form

$$g_2 = u_2 h_2, \tag{2.13}$$

where u_2 is an orthogonal matrix and h_2 a triangular matrix of the form

$$h_2 = \begin{pmatrix} h_{11} & 0 \\ h_{21} & h_{22} \end{pmatrix}.$$

Such a representation will be unique if we assume in addition that $h_{22} > 0$.

From this it follows immediately that the element $\hat{d}$ of the group $\hat{D}_m$ which satisfy the condition (2.12) can be represented, and moreover uniquely, in the form

$$\hat{d} = u\dot{d}, \tag{2.14}$$

where $u \in U_m$, $\dot{d} \in \dot{D}_m$.

Thus the elements k of the group $\hat{K}_m$ can be represented, and in fact uniquely, in the form

$$k = \zeta c u \dot{d}, \tag{2.15}$$

where $\zeta \in Z_m$, $c \in C_m$, $u \in U_m$ and $\dot{d} \in \dot{D}_m$.

In virtue of (2.1′) and (2.15), almost all elements g of the group G can be represented, and moreover uniquely, in the form

$$g = \zeta c u \dot{d} x, \tag{2.16}$$

where $\zeta \in Z_m$, $c \in C_m$, $u \in U_m$, $\dot{d} \in D_m$ and $x \in X_m$.

§3. The subgroups $S_{m,r}$ and $H_{m,r}$. Certain integral relations

A special role is played in the sequel by the subgroups $S_{m,r}$ and $H_{m,r}$ (or more briefly S_m and H_m) of the real group G_n, which we shall now consider.

1. The subgroup S_m. We consider in the group G the aggregate of all matrices s of the form

$$s = \zeta c u \quad (\zeta \in Z_m, c \in C_m, u \in U_m). \tag{3.1}$$

It may be seen immediately that these matrices form a subgroup of the group G, which we shall denote by S_m.

In view of (3.1) the elements $s = \| s_{pq} \|$ of the group S_m satisfy the conditions:

$$\left.\begin{array}{l} s_{pq} = 0 \quad \text{for} \quad p > q; \\ \text{for} \quad p \leq m \quad s_{pp} \quad \text{is an orthogonal matrix, multiplied by a scalar;} \\ \text{for} \quad p > m \quad s_{pp} \quad \text{is an arbitrary number.} \end{array}\right\} \tag{3.2}$$

Let us find an expression for a measure in the group S_m. We put, for brevity,

$$\det s_{pp} = \Lambda_p, \quad p = 1, \cdots, m + r. \tag{3.3}$$

Then

$$\Lambda_1 \Lambda_2 \cdots \Lambda_{m+r} = 1. \tag{3.4}$$

We put further, if $m > 0$,

$$\tilde{s}_{11} = \frac{1}{\sqrt{\Lambda_1}} s_{11}. \tag{3.5}$$

Then $\tilde{s}_{11}$ is an orthogonal matrix (if $m = 0$ we put $\tilde{s}_{11} = 1$). As parameters in s we take the parameters which define the matrices $\tilde{s}_{11}$ and s_{pq} $(p < q,\ q > 1)$.

Let us consider a left displacement $s' = s^{\circ} s$ by means of a matrix $s^{\circ} = \| s_{pq} \|$. This is in effect the mapping

$$\left.\begin{array}{l} \tilde{s}'_{11} = \tilde{s}^0_{11} \tilde{s}_{11}, \ s'_{12} = s^0_{11} s_{12} + s^0_{12} s_{22}, \cdots, \quad s'_{1,\, m+\tau} = \\ \qquad = s^0_{11} s_{1,m+\tau} + \cdots + s^0_{1,m+\tau} s_{m+\tau,\, m+\tau}, \\ s'_{22} = s^0_{22} s_{22}, \ s'_{23} = s^0_{22} s_{23} + s^0_{23} s_{33}, \ \ldots, \ s'_{2,\, m+\tau} = \\ \qquad = s^0_{22} s_{2,\, m+\tau} + \cdots + s^0_{2,m+\tau} s_{m+\tau,\, m+\tau} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ s'_{m+1,\, m+\tau} = s^0_{m+\tau,m+\tau}\, s_{m+\tau,\, m+\tau}. \end{array}\right\} \tag{3.6}$$

The determinant of this mapping is

$$\begin{aligned} \Delta^0_l = \left(\Lambda^0_1\right)^{r_2 + r_3 + \cdots + r_{m+\tau}} \left(\Lambda^0_2\right)^{r_2 + r_3 + \cdots + r_{m+\tau}} \left(\Lambda^0_3\right)^{r_3 + \cdots + r_{m+\tau}} \left(\Lambda^0_{m+\tau}\right)^{r_{m+\tau}} = \\ = \left(\Lambda^0_3\right)^{-r_2} \left(\Lambda^0_4\right)^{-r_2 - r_3} \ldots \left(\Lambda^0_{m+\tau}\right)^{-r_2 - r_3 - \cdots - r_{m+\tau-1}}. \end{aligned}$$

Hence a left-invariant measure in S_m is given by the formula

$$d\mu_l(s) = \frac{1}{|\Delta_l|} d\mu(\tilde{s}_{11}) \cdot \prod_{p=2}^{m+\tau} d\mu(s_{pp}) \cdot \prod_{p<q} d\mu(s_{pq}). \tag{3.7}$$

Here $d\mu(\tilde{s}_{11})$ is an invariant measure in the group of orthogonal matrices $\tilde{s}_{11}$ (in the case $m = 0$ this factor is absent), $d\mu(s_{pp})$ are invariant measures in the groups of matrices of the form s_{pp}, and $d\mu(s_{pq})$ is the product of the differentials of all elements of the matrices s_{pq}. In an entirely similar way we find that a right-invariant measure in S_m is given by the formula

$$d\mu_r(s) = \frac{1}{|\Delta_r|} d\mu(\tilde{s}_{11}) \cdot \prod_{p=2}^{m+\tau} d\mu(s_{pp}) \prod_{p<q} d\mu(s_{pq}), \tag{3.8}$$

where

$$\Delta_r = (\Lambda_2)^{r_1+r_1} (\Lambda_3)^{r_1+r_2+r_3} \dots (\Lambda_{m+\tau})^{r_1+r_2+\dots+r_{m+\tau}}.$$

Putting

$$\beta(s) = \frac{d\mu_l(s)}{d\mu_r(s)}, \tag{3.9}$$

we get

$$\beta(s) = |\Lambda_2|^{r_1+r_2} |\Lambda_3|^{r_1+2r_2+r_3} \dots |\Lambda_{m+\tau}|^{r_1+2r_2+\dots+2r_{m+\tau-1}+r_{m+\tau}}. \tag{3.10}$$

2. The subgroup H_m. We consider in the group G the aggregate of matrices h in the form

$$h = \dot{d}x \quad (\dot{d} \in \dot{D}_m, x \in X_m). \tag{3.11}$$

It is immediately apparent that these matrices form a subgroup of the group G, which we shall denote by H_m.

In virtue of (3.11) the elements $h = \| h_{pq} \|$ of the group H_m satisfy the conditions:

$$\left.\begin{array}{l} \text{for } p > q \ h_{pq} = 0; \text{ for } p \geq m \ h_{pp} \text{ is a second-order matrix of the form} \\ h_{pp} = \begin{pmatrix} h_{11}^{(p)} & 0 \\ h_{21}^{(p)} & h_{22}^{(p)} \end{pmatrix}, \text{ where } h_{22}^{(p)} > 0 \text{ and } \det h_{pp} = \pm 1; \\ \text{for } p = m+1, \dots, m+r-1 \ h_{pp} = 1 \text{ and } h_{m+r, m+r} = \pm 1. \end{array}\right\} \tag{3.12}$$

Reasoning as in the case of the group S_m, but taking into account that $|\det h_{pp}| = 1$ $(p = 1, \dots, m+r)$, we obtain the following result:

A right-invariant measure in the group H_m is given by the formula

$$d\mu_r(h) = \prod_{p=1}^{m} d\mu_r(h_{pp}) \prod_{p<q} d\mu(h_{pq}). \tag{3.13}$$

Here the $d\mu_r(h_{pp})$ are right-invariant measures in the groups of matrices of the form h_{pp}, $d\mu(h_{pq})$ the product of the differentials of all elements of the matrix h_{pq}.

3. Decomposition of elements of the group G. In view of (2.16) we deduce:

Almost all elements of the group G (with the exception of a manifold of lower dimensionality) can be represented, and moreover uniquely, in the form

$$g = sh, \tag{3.14}$$

where $s \in S_m$, $h \in H_m$.

4. An integral relation on the group H_m. We consider a function $f(h)$, summable on H_m. Since every element $h \in H_m$ can be represented uniquely in the form $d = \dot{d}x$, it follows that $f(h) = f(\dot{d}x)$ is a function of $\dot{d} \in \dot{D}_m$ and $x \in X_m$.

We obtain here a formula which reduces integration over H_m to repeated integration over $\dot{D}_m$ and X_m.

The connection between the elements of the matrices $h = \| h_{pq} \|$, $x = \| x_{pq} \|$ and $\dot{d} = [\dot{d}_1, \dot{d}_2, \cdots, \dot{d}_{m+r}]$ is expressed by the formulae

$$h_{pp} = \dot{d}_p, \quad h_{pq} = \dot{d}_p x_{pq} \quad \text{for} \quad p > q.$$

Taking into account that $| \det d_p | = 1$ we get

$$d\mu_r(h_{pp}) = d\mu_r(\dot{d}_p), \quad d\mu(h_{pq}) = d\mu(x_{pq})$$

(for fixed $\dot{d}_p$).

Taking into consideration the formulae (1.8), (1.20), and (3.13) for the measures in the groups in question we have:

$$\int f(h)\, d\mu_r(h) = \int d\mu(x) \int f(\dot{d}x)\, d\mu_r(\dot{d}). \tag{3.15}$$

5. An integral relation on the group G. In the group G there exists the two-sided-invariant measure $d\mu(g)$. We consider a function $f(g)$, summable on G. Since almost every element $g \in G$ can be represented uniquely in the form $g = sh$, it follows that $f(g) = f(sh)$ constitutes a function of $s \in S_m$ and $h \in H_m$.

We obtain here a formula which reduces integration over G to repeated integration over S_m and H_m.

In as much as the equation $g = sh$ establishes an analytic dependence between the parameters, in terms of which the measure in the group G is expressed, and the corresponding parameters in S_m and H_m, there must hold a formula

$$\int f(g)\, d\mu(g) = \int d\mu_r(h) \int f(sh)\, \omega(s, h)\, d\mu_l(s), \tag{3.16}$$

where $\omega(s, h)$ is a certain positive function of the variables s and h.

It is easy to see, however, that the function $\omega(s, h)$ is in fact a constant c, depending on the normalisation of the measures $d\mu(g)$, $d\mu_l(s)$ and $d\mu_r(h)$.

For by the invariance of the measure $d\mu(g)$

$$\int f(g)\, d\mu(g) = \int f(s_0 g)\, d\mu(g).$$

Hence, by (3.16) and the left invariance of the measure $d\mu_l(s)$, we have:

$$\int d\mu_r(h) \int f(sh)\, \omega(s, h)\, d\mu_l(s) = \int d\mu_r(h) \int f(s_0 sh)\, \omega(s, h)\, d\mu_l(s) =$$

$$= \int d\mu_r(h) \int f(sh)\, \omega(s_0^{-1} s, h)\, d\mu_l(s).$$

In view of the arbitrary character of the function $f(sh)$ this means that

$$\omega(s, h) = \omega(s_0^{-1} s, h),$$

i.e., the function $\omega(s, h)$ does not depend on s. Similarly, by the invariance of the invariance of the measure $d\mu(g)$ and the *right* invariance of the measure $d\mu_r(h)$, we deduce from the formula (3.16) that

$$\omega(s, h) = \omega(s, h h_0^{-1}),$$

i.e., the function $\omega(s, h)$ does not depend on h. It is thus proved that $\omega(s, h)$ is a constant c, and consequently

$$\int f(g)\,d\mu(g) = c\int d\mu_r(h)\int f(sh)\,d\mu_l(s). \tag{3.17}$$

6. **A transformation formula for the measure in H_m.** Using (3.17) we shall now find how the measure $d\mu_r(h)$ transforms under the mapping $h_1 = h\bar{g}$.

We show to begin with that in the group S_m there holds the formula

$$\int f(ss_1)\,d\mu_l(s) = \beta^{-1}(s_1)\int f(s)\,d\mu_l(s). \tag{3.18}$$

For passing from the left-invariant measure $d\mu_l(s)$ to the right-invariant measure $d\mu_r(s)$ in accordance with (3.9) we have:

$$\int f(ss_1)\,d\mu_l(s) = \int f(ss_1)\,\beta(s)\,d\mu_r(s) = \int f(s)\,\beta(ss_1^{-1})\,d\mu_r(s).$$

But the function $\beta(s)$, given by (3.10), has the following properties:

$$\beta(s_1s_2) = \beta(s_1)\,\beta(s_2), \quad \beta(s^{-1}) = \beta^{-1}(s). \tag{3.19}$$

Hence

$$\int f(s)\,\beta(ss_1^{-1})\,d\mu_r(s) =$$
$$= \int f(s)\,\beta^{-1}(s_1)\,\beta(s)\,d\mu_r(s) = \beta^{-1}(s_1)\int f(s_1)\,d\mu_l(s),$$

and the equation (3.18) is thereby proved.

Substituting in (3.17) a function of the form $x(g) = f(h)\,\phi(s)$, we obtain:

$$\int x(g)\,d\mu(g) = c\int f(h)\,d\mu_r(h)\int \varphi(s)\,d\mu_l(s). \tag{3.20}$$

Let g_0 be an arbitrary element of G. We put

$$hg_0 = s_1h_1, \text{ and so } h\bar{g}_0 = h_1. \tag{3.21}$$

Then $gg_0 = shg_0 = ss_1h_1$ and $x(gg_0) = f(h\bar{g}_0)\,\phi(ss_1)$. By the invariance of the measure $d\mu(g)$ we have:

$$\int x(g)\,d\mu(g) = \int x(gg_0)\,d\mu(g),$$

whence, by (3.17) and (3.18),

$$\int f(h)\,d\mu_r(h)\int \varphi(s)\,d\mu_l(s) = \int f(h\bar{g}_0)\,d\mu_r(h)\int \varphi(ss_1)\,d\mu_l(s) =$$
$$= \int f(h\bar{g}_0)\,\beta^{-1}(s_1)\,d\mu_r(h)\int \varphi(s)\,d\mu_l(s).$$

Consequently,

$$\int f(h)\,d\mu_r(h) = \int f(h\bar{g}_0)\,\beta^{-1}(s_1)\,d\mu_r(h), \tag{3.22}$$

where the element s_1 is defined by the equation

$$hg_0 = s_1h_1. \tag{3.23}$$

Re-writing the equation (3.22) in the form

$$\int f(h\bar{g}_0)\, d\mu_r(h\bar{g}_0) = \int f(h\bar{g}_0)\beta^{-1}(s_1)\, d\mu_r(h),$$

we obtain the desired formula for the transformation of the measure in H_m :

$$\frac{d\mu_r(h\bar{g}_0)}{d\mu_r(h)} = \beta^{-1}(s_1). \tag{3.24}$$

§4. Certain transitive manifolds in Z

We consider the subgroup Z of complex triangular matrices $z = \| z_{ij} \|$, where $z_{ii} = 1$ and $z_{ij} = 0$ for $i < j$ $(i, j = 1, \cdots, n)$. According to [1], almost every element g of the complex group $\mathfrak{G}$ can be represented, and moreover uniquely, in the form of a product $g = kz$, where $z \in Z$, and the matrix k has zeros under its principal diagonal. In view of this, every element g of the group $\mathfrak{G}$ determines a mapping $z \to z_1 = z\bar{g}$ of the group Z, where z_1 is determined from the relation $zg = kz_1$.

In this section we describe certain manifolds in the subgroup Z, which are transitive with respect to mappings of Z by elements of the real group G.

1. The manifolds $\dot{Z}_{m,r}$, $\dot{Z}^{+}_{m,r}$ **and** $\dot{Z}^{-}_{m,r}$. We denote by $\dot{Z}_{m,r}$ the aggregate of matrices $\dot{z}$ from $D'_{m,r}$ of the form

$$\dot{z} = [v_1, v_2, \cdots, v_m, 1, \cdots, 1], \tag{4.1}$$

where the v_p are second-order matrices of the form

$$v_p = \begin{pmatrix} 1 & 0 \\ z_p & 1 \end{pmatrix}, \quad \operatorname{Im} z_p \neq 0 \quad (p = 1, 2, \ldots, m). \tag{4.2}$$

For $m = \frac{n}{2}$ we also introduce the sets $\dot{Z}^{+}_{m,0}$ and $\dot{Z}^{-}_{m,0}$. $\dot{Z}^{+}_{m,0}$ in the set of matrices $\dot{z}^{+}$ from $\dot{Z}_{m,0}$ of the form

$$\dot{z}^{+} = [v_1, v_2, \cdots, v_m], \tag{4.1a}$$

where the v_p are matrices of the form (4.2), satisfying the condition

$$\operatorname{Im} z_1 \cdot \operatorname{Im} z_2 \cdots \operatorname{Im} z_m > 0. \tag{4.3}$$

Similarly $\dot{Z}^{-}_{m,0}$ is the set of matrices $\dot{z}^{-}$ from $\dot{Z}_{m,0}$ of the form (4.1a), where the v_p are second-order matrices of the form (4.2), satisfying the condition

$$\operatorname{Im} z_1 \cdot \operatorname{Im} z_2 \cdots \operatorname{Im} z_m < 0. \tag{4.3a}$$

For $m \neq \frac{n}{2}$, the elements of the set $\dot{Z}_{m,r}$ can be translated one into another by mappings of the group Z by means of elements $g \in G$. For $m = \frac{n}{2}$ the same statement is true in regard to the sets $\dot{Z}^{+}_{m,0}$ and $\dot{Z}^{-}_{m,0}$.

Proof. If v_p and v'_p are two second-order matrices of the form (4.2) then, as is known, there exists a non-singular real second-order matrix g_p such that

$$v_p g_p = k_p v'_p, \tag{4.4}$$

where k_p is a matrix of the form

$$k_p = \begin{pmatrix} k_{11}^{(p)} & k_{12}^{(p)} \\ 0 & k_{22}^{(p)} \end{pmatrix}, \tag{4.5}$$

i.e., $v_p \bar{g}_p = v_p'$. Moreover, if the numbers z_p and z_p' belong to one and the same half-plane $\operatorname{Im} z > 0$ or $\operatorname{Im} z < 0$, then $\det g_p > 0$. In the opposite case $\det g_p < 0$. We may always assume that $|\det g_p| = 1$. Hence we find easily that for $m \neq \frac{n}{2}$ the matrix (4.1) transforms into a matrix

$$\dot{z}' = [v_1', v_2', \cdots, v_m', 1, \cdots, 1]$$

by means of a real unimodular matrix $g \in D_{m,r}$ of the form

$$g = [g_1, g_2, \cdots, g_m, c_{m+1}, \cdots, c_{m+r}], \tag{4.6}$$

where $c_{m+1}, \cdots, c_{m+r}$ are arbitrary numbers satisfying the condition

$$c_{m+1} \cdots c_{m+r} \cdot \det g_1 \det g_2 \cdots \det g_m = 1.$$

For $m = \frac{n}{2}$ the corresponding matrix g,

$$g = [g_1, g_2, \cdots, g_m], \tag{4.6a}$$

will be unimodular if and only if the signs of the expressions

$$\operatorname{Im} z_1 \cdot \operatorname{Im} z_2 \cdots \operatorname{Im} z_m$$

for the matrices $\dot{z}$ and $\dot{z}'$ coincide, i.e., if and only if the matrices $\dot{z}$ and $\dot{z}'$ belong simultaneously to $\dot{Z}^+_{m,0}$ or $\dot{Z}^-_{m,0}$.

Definition. For $m \neq \frac{n}{2}$, we denote by $Z_{m,r}$ that manifold in Z which is transitive with respect to mappings by elements $g \in G$, and which contains the manifold $\dot{Z}_{m,r}$. If n is even, we denote by $Z^+_{n/2,0}$ and $Z^-_{n/2,0}$ the transitive manifolds in Z which contain the manifolds $\dot{Z}^+_{n/2,0}$ and $\dot{Z}^-_{n/2,0}$, respectively.

2. Analytic characterisation of the manifolds $Z_{m,r}$, $Z^+_{n/2,0}$ and $Z^-_{n/2,0}$. *Every matrix z from $Z_{m,r} (m \neq \frac{n}{2})$ can be represented, and moreover uniquely, in the form*

$$z = \dot{z}x \quad (\dot{z} \in \dot{Z}_{m,r}, x \in X_{m,r}). \tag{4.7}$$

For $m = \frac{n}{2}$ every matrix z^+ from $Z^+_{m,0}$ (z^- from $Z^-_{m,0}$) can be represented, and moreover uniquely, in the form

$$z^+ = \dot{z}^+ x \quad (z^- = \dot{z}^- x), \tag{4.7a}$$

where $\dot{z}^+ \in \dot{Z}^+_{m,0} (\dot{z}^- \in Z^-_{m,0})$ and $x \in X_{m,0}$. Conversely, all matrices which can be represented in these ways belong to the corresponding transitive manifolds.*

Proof. Let us take the case $m \neq \frac{n}{2}$. To begin with, matrices of the form $\dot{z}x$, where $\dot{z} \in \dot{Z}_m$ and $x \in X_m$, belong to the manifold Z_m. This follows immediately from the transitivity of the manifold Z_m and the obvious fact that the elements $\dot{z}$ and $\dot{z}x$ belong to one and the same transitive manifold.

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* We shall as a rule omit the index r in the notation for the manifolds.

Conversely, let us fix in the manifold $\dot{Z}_m$ a matrix

$$z_0 = [v_1^0, \cdots, v_m^0, 1, \cdots, 1], \tag{4.8}$$

where the v_p^0 are matrices of the form

$$v_p^0 = v_1^0 = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \quad p = 1, \ldots, m. \tag{4.9}$$

We shall prove that an arbitrary matrix $z_0\overline{g}$, where $g \in G$ (provided only that this is definite in Z), can be represented in the form $z_0\overline{g} = \dot{z}x$. First of all, almost every matrix $g \in G$ can be represented in the form $g = \hat{k}x$, where $\hat{k} \in \hat{K}_m$, $x \in X_m$. Exceptions are formed only by matrices g for which at any rate one of the minors $g_{r_1 + \cdots + r_p + 1}$ (see §2) vanishes. Since, by hyposhesis, the matrix $z_0\overline{g}$ is definite, the minors $(z_0 g)_s$ of the matrix $z_0 g$ $(s = 2, \cdots, n)$ must be different from zero. We remark that the matrix $z_0 g$ is obtained from the matrix g by adding to every line with the number $2p$ $(p = 1, \cdots, m)$ the previous line, multiplied by i. From this it follows that

$$g_{r_1+\cdots+r_p+1} = (z_0 g)_{r_1+\cdots+r_p+1},$$

and therefore all the minors $g_{r_1 + \cdots + r_p + 1}$ are different from zero. Thus the matrix g can be put in the form $g = \hat{k}x$.

Let us consider the matrix $\hat{k} = \| k_{pq} \|$. The matrices k_{pp} $(p = 1, \cdots, m)$ are non-singular, and therefore the matrices $v_p^0 k_{pp}$ can be represented in the form

$$v_p^0 k_{pp} = k_p v_p,$$

where k_p and v_p are matrices of the form

$$k_p = \begin{pmatrix} k_{11}^{(p)} & k_{12}^{(p)} \\ 0 & k_{22}^{(p)} \end{pmatrix}, \quad v_p = \begin{pmatrix} 1 & 0 \\ z_p & 1 \end{pmatrix}, \quad \operatorname{Im} z_p \neq 0.$$

From this it follows immediately that the matrix $z_0\hat{k}$ can be put in the form $z_0\hat{k} = k\dot{z}$, where $k \in K$, $\dot{z} \in \dot{Z}_m$. Thus we have $z_0 g = k\dot{z}x$, i.e.,

$$z_0\overline{g} = \dot{z}x.$$

The uniqueness of the representation in the form $z = \dot{z}x$ follows immediately from the fact that the group D_m' which contains the whole of the manifold $\dot{Z}_m$ has as its intersection with the group X_m only the unit matrix.

The above arguments all retain their force for the manifolds Z_m^+ and Z_m^-. In the case of the manifold Z_m^- we have to start from an element z_0 of the form (4.8), where

$$v_1^0 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \text{ and } v_p^0 = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \quad (p = 2, \ldots, m). \tag{4.9a}$$

3. Characterisation of the manifold Z_m by means of the subgroup H_m. In accordance with the results of §3, every element $g \in G$ which can be represented in the form

$$g = \hat{k}x \quad (\hat{k} \in \hat{K}_m, x \in X_m),$$

can also be represented, and moreover uniquely, in the form $g = sh$, where $s \in S_m$, $h \in H_m$. Every element $h \in H_m$ can in turn be represented, and moreover uniquely, in the form $h = \dot{d}x$.

We shall start from a fixed element z_0 of the manifold Z_m, defined by the formulae (4.8) and (4.9).

With the element h of H_m we associate an element $z_0\overline{h}$ of Z_m:

$$h \to z_0\overline{h}. \tag{4.10}$$

It is easy to see that the formula (4.10) constitutes a one-to-one mapping of the manifold H_m on Z_m. For if $h = \dot{d}x$ then, obviously,

$$\dot{d}x \to (z_0\overline{\dot{d}})x. \tag{4.10a}$$

At the same time the correspondence $\dot{d} \to z_0\overline{\dot{d}}$ constitutes a one-to-one mapping of the subgroup $\dot{D}_m$ on $\dot{Z}$ (cf. §1, sub-section 5).

We show that the mappings $h \to h\overline{g}$ of the group H_m by elements g of the group G:

$$h\overline{g} \text{ is an element } h_1 \text{ of } H_m \text{ such that } hg = sh_1, \tag{4.11}$$

coincide with mappings in Z_m. In other words, elements from H_m and Z_m which correspond to one another are transformed by any $g \in G$ again into corresponding elements.

We observe that to the unit element e in H_m there corresponds in Z_m the element z_0. Our assertion will be proved completely if we show that to the element $e\overline{g}$ from H_m there corresponds the element $z_0\overline{g}$ from Z_m, where g is any element from G.

Let $g = sh$. Then, obviously, $e\overline{g} = h$. On the other hand

$$\overline{z}_0g = (z_0\overline{s})\overline{h}.$$

But the matrix s can be represented in the form $s = \zeta cu$, where $\zeta \in Z_m$, $c \in C_m$, $u \in U_m$. It is easy to see that $z_0\overline{\zeta} = z_0$, $z_0\overline{c} = z_0$ and $z_0\overline{u} = z_0$. Hence $z_0\overline{s} = z_0$, and so $z_0\overline{g} = z_0\overline{h}$, and our statement is proved.

We note that S_m forms a stationary subgroup for the element z_0.

For $m = \frac{n}{2}$ each of the manifolds Z_m^+ and Z_m^- can be characterised in an analogous fashion by means of the corresponding subgroup H_m.

The operator of the representations will have effect in spaces of functions $f(\dot{z}x) = f(z_1, \cdots, z_m, x)$ on the manifold Z_m defined above.

§5. Analyticity of the functions $f(z_1, \cdots, z_m, x)$

In accordance with [1], the representations of the complex unimodular group $\mathfrak{G}$ are formed in the space of functions, defined in the manifold Z' of right cosets of the group $\mathfrak{G}$ with respect to the subgroup K of complex triangular matrices. In a similar way they may be given in the space of cosets of the group $\mathfrak{G}$ with respect to any subgroup K_1 which is conjugate to the group K.

In the case of the real unimodular group G it is necessary to make a distinction between the subgroup K of triangular matrices and the subgroups $K' = g_1^{-1} K g_1$ of the complex group $\mathfrak{G}$ which are conjugate to it. For the intersections of these subgroups with the group G of real matrices may be distinct, and therefore are bound to lead us to distinct representations of the group G of real matrices.

The representations which correspond to a given subgroup $K' = g_1^{-1} K g_1$ are formed in the space of functions on elements of the manifold of right cosets of the group G with respect to its intersection $G \cap K'$ with the subgroup K'. However, this space must consist, in general, not of all functions but only of functions which are analytic in certain of the variables.

Let us explain the appearance here of analytic functions. We first return to the complex group $\mathfrak{G}$. The space of functions which are constant on right cosets of the group $\mathfrak{G}$ with respect to its subgroup K may also be characterised by the condition that these functions should be constant under infinitely small displacements, corresponding to multiplication on the left of elements of the group $\mathfrak{G}$ by infinitesimal elements K. Since the infinitely small displacements are given by operators X_i, it follows that these functions satisfy the equations

$$X_i f = 0, \tag{5.1}$$

where the X_i are Lie operators corresponding to infinitesimal elements from K.

Let us now pass to the real group. We shall consider functions $f(g)$ on the group G of real matrices. Let K_B be the subgroup of real triangular matrices, and $K' = g_1^{-1} K_B g_1$ some subgroup of the group $\mathfrak{G}$ of complex matrices which is conjugate to K_B. We first assume that the matrix g_1, by means of which the subgroup K' is defined, is real. Then K' is a subgroup of the real group G. Let us require, as in the case of the complex group, that the function $f(g)$ should be constant under infinitely small displacements corresponding to left multiplication of elements of the group G by infinitesimal elements from K'. This requirement may be expressed, as in the case of the complex group, in the form of a certain system of linear differantial equations $X_i f = 0$, which the functions $f(g)$ are to satisfy. The coefficients of these equations will be expressible analytically in terms of the elements of the matrix g_1. We can consequently write down formally the Lie operators also for the case when the matrix g_1 is no longer real, and require that the functions $f(g)$ should satisfy the corresponding system of differential equations $X_i f = 0$.

In the case when the matrix g_1 is real the resusting system of differential equations is hyperbolic, as also in the case of the complex group. As its characteristics we shall have precisely the right cosets of the real group G with respect to the subgroup K'. If, however, the matrix g_1 is complex, the corresponding system of equations for $f(g)$ will no longer be hyperbolic. Its solutions will be functions which are constant on the cosets of the group G with respect to the corresponding subgroup (in fact with respect to the subgroup $G \cap g_1^{-1} K g_1$, where K is the subgroup of complex triangular matrices) and analytic in certain of the parameters of the group. The constancy of the functions in certain parameters of the group thus changes to the requirement of analyticity, under the transition from the complex group to the real group.

In what follows we shall consider the subgroup

$$K' = z_0^{-1} K_B z_0,$$

where z_0 is the element of the manifold Z_m defined by (4.8) and (4.9). In this section we show that the solutions of the corresponding system of differential equations will be functions which are constant on the right cosets of the group G with respect to the subgroup $z_0^{-1} K z_0 \cap G$, where K is the group of complex triangular matrices, and which are analytic in certain parameters. To identify these parameters we remark that the subgroup $z_0^{-1} K z_0 \cap G$ is a stationary subgroup for the element z_0 of the manifold Z_m. Hence the manifold of right cosets of the group G with respect to this subgroup can be identified with the manifold Z_m, and one can define functions on this latter manifold. We show here that these functions must be analytic in each, in fact, of the complex parameters $z_1, z_2, \cdots, z_m$ of the manifold Z_m (see §4).

We give first an account on the elements of the group K_B of real triangular matrices. Since this group is a subgroup of the group $\hat{K}_m$ of triangular matrices of boxes, its elements, in accordance with §2, can be expressed (and moreover uniquely) in the form

$$k = \zeta c \hat{d}, \tag{5.2}$$

where $\zeta \in Z_m$, $c = [c_1, c_2, \cdots, c_{m+r}] \in C_m$, and the matrix $\hat{d} = [\hat{d}_1, \hat{d}_2, \cdots, \hat{d}_{m+r}] \in \hat{D}_m$ and satisfies the condition (2.12):

$$\hat{d}_{m+1} = \cdots = \hat{d}_{m+r-1} = 1.$$

Here the matrices $\hat{d}_p$ $(p = 1, \cdots, m)$ have the form:

$$\hat{d}_p = \begin{pmatrix} k_{11}^{(p)} & k_{12}^{(p)} \\ 0 & k_{22}^{(p)} \end{pmatrix}, \quad |\det \hat{d}_p| = 1. \tag{5.3}$$

We proceed to determine the form of the matrix $z_0^{-1}kz_0$. We remark first of all that $z_0^{-1}\zeta z_0$ is a matrix belonging to the group Z'_m of complex triangular matrices of boxes, and that $z_0^{-1}cz_0 = c$. By direct calculation we find further that

$$z_0^{-1}\,\hat{d}z_0 = \bar{d} = [\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_{m+\tau}], \tag{5.4}$$

where

$$\bar{d}_p = \begin{pmatrix} k_{11}^{(p)} + ik_{12}^{(p)} & k_{12}^{(p)} \\ k_{12}^{(p)} + i\,(k_{23}^{(p)} - k_{11}^{(p)}) & k_{22}^{(p)} - ik_{12}^{(p)} \end{pmatrix} \quad (p \leqslant m). \tag{5.5}$$

In view of this we conclude the following:

Every element k' of the subgroup K' can be expressed in the form

$$k' = \zeta' c\bar{d}, \tag{5.6}$$

where ζ' is an element of the group $z_0^{-1}Z_m z_0$ which is a subgroup of the group Z'_m, $c \in C_m$, $\bar{d} \in D'_m$ and is defined by (5.4) and (5.5).

By §2, almost every element g of the real group G can be expressed, and moreover uniquely, in the form

$$g = \zeta c\hat{d}x, \tag{5.7}$$

where $\zeta \in Z_m$, $c \in C_m$, $x \in X_m$, and $\hat{d} = [\hat{d}_1; \hat{d}_2, \cdots, \hat{d}_{m+r}] \in \hat{D}_m$ and satisfies the condition (2.12):

$$\hat{d}_{m+1} = \cdots = \hat{d}_{m+r-1} = 1.$$

Hence we may regard any function, defined on the group G, as a function of $\zeta \in Z_m$, $c \in C_m$, $\hat{d} \in D_m$, $x \in X_m$, and write

$$f(g) = (\zeta c\hat{d}x). \tag{5.8}$$

In order to form the differential equations which the function $f(g)$ satisfies, we must consider the infinitesimal elements of the subgroups $z_0^{-1}Z_m z_0$, C_m and the subgroups $\bar{D}$ of elements $\bar{d}$ defined by (5.4) and (5.5). It is easy to see, however, that the system of differential equations $X_i^{(z)}f = 0$, which corresponds to the infinitesimal elements of $z_0^{-1}Z_m z_0$, is equivalent to the system of differential equations which corresponds to the infinitesimal elements of the subgroup Z_m itself. Hence the solutions of this system of equations will be functions constant on the cosets of the group G with respect to Z_m. Consequently we may write:

$$f(\zeta c\hat{d}x) = f(c\hat{d}x). \tag{5.9}$$

In a similar way consideration of the differential equations $X_i^{(C)}f = 0$, corresponding to infinitesimal elements of C_m, shows that the functions $f(c\hat{d}x)$ do not depend on $c \in C_m$, and so

$$f(c\hat{d}x) = f(\hat{d}x). \tag{5.10}$$

Let us now consider infinitesimal elements from $\bar{D}$, and form the associated system of differential equations for the functions f.

Inserting in (5.5)

$$k_{11}^{(p)} = 1 + dt_1^{(p)}, \quad k_{12}^{(p)} = dt_2^{(p)},$$

we obtain the infinitesimal element

$$\delta \bar{d} = [\delta \bar{d}_1, \ \delta \bar{d}_2, \ldots, \delta \bar{d}_m, \ 0, \ldots, 0], \tag{5.11}$$

where the matrices $\delta \bar{d}_p$ $(p = 1, \cdots, m)$ have the form

$$\delta \bar{d}_p = \begin{pmatrix} dt_1^{(p)} + i dt_2^{(p)} & dt_2^{(p)} \\ dt_2^{(p)} - 2i dt_1^{(p)} & - dt_1^{(p)} - i dt_2^{(p)} \end{pmatrix}. \tag{5.12}$$

Let

$$\hat{d} = [\hat{d}_1, \ \hat{d}_2, \ldots, \hat{d}_{m+\tau}], \tag{5.13}$$

where

$$\hat{d}_p = \begin{pmatrix} \alpha_p & \beta_p \\ \gamma_p & \delta_p \end{pmatrix} \quad (p \leqslant m). \tag{5.14}$$

Then

$$\delta \bar{d} \cdot \hat{d} = [\delta \bar{d}_1 \cdot \hat{d}_1, \ \delta \bar{d}_2 \cdot \hat{d}_2, \ldots, \delta \bar{d}_m \cdot \hat{d}_m, \ 0, \ldots, 0], \tag{5.15}$$

where

$$\delta \bar{d}_p \cdot \hat{d}_p =$$

$$= \begin{pmatrix} \alpha_p dt_1^{(p)} + (\gamma_p + i\alpha_p)\, dt_2^{(p)} & \beta_p dt_1^{(p)} + (\delta_p + i\beta_p)\, dt_2^{(p)} \\ -(\gamma_p + 2i\alpha_p)\, dt_1^{(p)} - i\,(\gamma_p + i\alpha_p)\, dt_2^{(p)} & -(\delta_p + 2i\beta_p)\, dt_1^{(p)} - i\,(\delta_p + i\beta_p)\, dt_2^{(p)} \end{pmatrix}. \tag{5.16}$$

The Lie operators which correspond to the infinitesimal displacements $dt_1^{(p)}$ and $dt_2^{(p)}$ have accordingly the following form:

$$X_1^{(p)} f = \frac{\partial f}{\partial \alpha_p} \alpha_p + \frac{\partial f}{\partial \beta_p} \beta_p + \frac{\partial f}{\partial \gamma_p} (-\gamma_p - 2i\alpha_p) + \frac{\partial f}{\partial \delta_p} (-\delta_p - 2i\beta_p),$$

$$X_2^{(p)} f = \frac{\partial f}{\partial \alpha_p} (\gamma_p + i\alpha_p) + \frac{\partial f}{\partial \beta_p} (\delta_p + i\beta_p) - i \frac{\partial f}{\partial \gamma_p} (\gamma_p + i\alpha_p) - i \frac{\partial f}{\partial \delta_p} (\delta_p + i\beta_p).$$

We require that the functions $f(\hat{d}x)$ should satisfy the differential equations $X_1^{(p)} f = 0$ and $X_2^{(p)} f = 0$, i.e., that there should hold the equations:

$$\frac{\partial f}{\partial \alpha_p} \alpha_p + \frac{\partial f}{\partial \beta_p} \beta_p + \frac{\partial f}{\partial \gamma_p} (-\gamma_p - 2i\alpha_p) + \frac{\partial f}{\partial \delta_p} (-\delta_p - 2i\beta_p) = 0 \tag{5.17}$$

and

$$\frac{\partial f}{\partial \alpha_p} (\gamma_p + i\alpha_p) + \frac{\partial f}{\partial \beta_p} (\delta_p + i\beta_p) - i \frac{\partial f}{\partial \gamma_p} (\gamma_p + i\alpha_p) - i \frac{\partial f}{\partial \delta_p} (\delta_p + i\beta_p) = 0, \tag{5.18}$$

where $p = 1, 2, \cdots, m$. Subtracting the equation (5.17), multiplied by i, from the equations (5.18), we get:

$$\frac{\partial f}{\partial \alpha_p} \gamma_p + \frac{\partial f}{\partial \beta_p} \delta_p - \frac{\partial f}{\partial \gamma_p} \alpha_p - \frac{\partial f}{\partial \delta_p} \beta_p = 0. \tag{5.19}$$

The system of equations for the characteristics of the equation (5.19) has the form:

$$\frac{d\alpha_p}{\gamma_p} = \frac{d\beta_p}{\delta_p} = \frac{d\gamma_p}{-\alpha_p} = \frac{d\delta_p}{-\beta_p}\,. \tag{5.20}$$

It is easy to find the first integrals of the system:

$$\alpha_p\delta_p - \beta_p\gamma_p = C_1,\ \alpha_p\beta_p + \gamma_p\delta_p = C_2 \text{ and } \beta_p^2 + \delta_p^2 = C_3. \tag{5.21}$$

Since furthermore the elements α_p, β_p, γ_p, and δ_p of the matrix $\hat{d}_p$ satisfy the condition

$$\alpha_p\delta_p - \beta_p\gamma_p = \pm 1,$$

it follows by the equations (5.19) that the functions $f(\hat{d}x)$ are functions of the variables

$$x_p = \frac{\alpha_p\beta_p + \gamma_p\delta_p}{\beta_p^2 + \delta_p^2}, \quad y_p = \frac{\alpha_p\delta_p - \beta_p\gamma_p}{\beta_p^2 + \delta_p^2} \tag{5.22}$$

and the elements of the matrix x.

We now explain the significance of the variables x_p and y_p. For this we put

$$z = z_0\bar{\hat{d}}\,. \tag{5.23}$$

Then, as is easily seen,

$$z = [v_1, v_2, \cdots, v_m, 1, \cdots, 1] \in \dot{Z}_m, \tag{5.24}$$

where

$$v_p = \begin{pmatrix} 1 & 0 \\ z_p & 1 \end{pmatrix}, \tag{5.25}$$

and furthermore

$$z_p = \frac{i\alpha_p + \gamma_p}{i\beta_p + \delta_p} = \frac{\alpha_p\beta_p + \gamma_p\delta_p}{\beta_p^2 + \delta_p^2} + i\,\frac{\alpha_p\delta_p - \beta_p\gamma_p}{\beta_p^2 + \delta_p^2} = x_p + iy_p. \tag{5.26}$$

Thus the numbers $z_p = x_p + iy_p$ constitute parameters which correspond to the element $\hat{d}x$ in the manifold Z_m.

As in (2.14), we represent the element $\hat{d}$ in the form $\hat{d} = u\dot{d}$, where $u \in U_m$, $\dot{d} \in \dot{D}_m$. Since $z_0\bar{u} = z_0$, and since $z_0\hat{d} = z_0\dot{d}$, the function $f(\hat{d}x) = f(u\dot{d}x)$ does not depend on u.

We deduce:

The functions $f(g)$, which are defined on G and which satisfy the differential equations $X_i^{(Z)}f = 0$, $X_i^{(C)}f = 0$ and (5.19), are functions which are constant on the right cosets of the group G with respect to the group S_m. In accordance with the results of §4, one may regard them as functions

$$f(\dot{z}x) = f(z_1, z_2, \cdots, z_m, x)$$

defined on the manifold Z_m.

We now return to the equations (5.18). We show that *from the equations* (5.18) *there follows the analyticity of the functions* $f(z_1, z_2, \cdots, z_m, x)$ *in each of the complex variables* z_p $(p = 1, \cdots, m)$ *for fixed values of the remaining variables.*

For this we introduce the operators $\partial/\partial z$ and $\partial/\partial \bar{z}$. If z is a complex variable $x + iy$, then by definition

$$\partial / \partial z = {}^1/_2(\partial / \partial x - i\partial / \partial y), \quad \partial / \partial \bar{z} = {}^1/_2 (\partial / \partial x + i\partial / \partial y). \tag{5.27}$$

If $f(z)$ is a function of the complex variable z, then the condition for it to be analytic will be

$$\frac{\partial f}{\partial \bar{z}} = 0. \tag{5.28}$$

We are therefore obliged to show that from the equations (5.18) it follows that

$$\frac{\partial f}{\partial \bar{z}_p} = 0 \quad (p = 1, \cdots, m) \tag{5.29}$$

Multiplying the equation (5.18) by i and grouping its terms we may re-write this equation in the form

$$\left(i \frac{\partial f}{\partial \alpha_p} + \frac{\partial f}{\partial \gamma_p}\right)(i\alpha_p + \gamma_p) + \left(i \frac{\partial f}{\partial \beta_p} + \frac{\partial f}{\partial \delta_p}\right)(i\beta_p + \delta_p) = 0. \tag{5.30}$$

If we put

$$i\alpha_p + \gamma_p = z'_p, \quad i\beta_p + \delta_p = z''_p, \tag{5.31}$$

then (5.30) takes the form

$$\frac{\partial f}{\partial \bar{z}'_p} z'_p + \frac{\partial f}{\partial \bar{z}''_p} z''_p = 0. \tag{5.32}$$

It remains for us to show that the left-hand side of this equation is proportional to $\partial f/\partial \bar{z}_p$. But this follows immediately from the following assertion:

If $f(z)$ is a function of the complex variable $z = x + iy$ and $z = z_1/z_2$, $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then

$$\frac{\partial f}{\partial \bar{z}_1} z_1 + \frac{\partial f}{\partial \bar{z}_2} z_2 = 2i \frac{z_2}{\bar{z}_2} \cdot \mathrm{Im}\, z \cdot \frac{\partial f}{\partial \bar{z}}. \tag{5.33}$$

All the considerations of this section remain in force for the case $m = \frac{n}{2}$ as well. In order to arrive at the manifold $Z^-_{n/2}$ we need only start from an element z_0 defined the formulae (4.8) and (4.9).

We remark that for $m \neq \frac{n}{2}$ the functions $f(z_1, z_2, \cdots, z_m, x)$ are defined in respect of each z_p (for fixed values of the remaining variables) alike in the upper half-plane $\mathrm{Im}\, z_p > 0$ and in the lower half-plane $\mathrm{Im}\, z_p < 0$. In view of what has been proved, these functions are analytic in each of the half-plane referred to.

In the case $m = \frac{n}{2}$ the functions $f(z_1, z_2, \cdots, z_m, x)$ are defined in respect of each z_p (for fixed values of the remaining variables) only in one of the half-planes $\operatorname{Im} z_p > 0$ and $\operatorname{Im} z_p < 0$, to be determined according to the sign of the expression

$$\operatorname{Im} z_1 \cdots \operatorname{Im} z_{p-1} \operatorname{Im} z_{p+1} \cdots \operatorname{Im} z_m.$$

In this half-plane these functions are analytic in z_p.

§6. Description of the representations of the principal nondegenerate series

We shall construct the representations of a principal nondegenerate series $d_{m,r}$ in the space $\mathfrak{H}_m$ of functions $f(z_1, z_2, \cdots, z_m, x)$ which are defined on the manifold Z_m and analytic, in accordance with §5, in each of the parameters $z_1, z_2, \cdots z_m$.

By §4, we can identify the manifold Z_m with the group H_m and write

$$f(z_1, z_2, \cdots, z_m, x) = f(h) \quad (h \in H_m).$$

We shall define a scalar product in the space $\mathfrak{H}_m$ in the form

$$(f_1, f_2) = \int f_1(h) \overline{f_2(h)}\, \omega(h)\, d\mu_r(h), \tag{6.1}$$

where $\omega(h)$ is a positive function of h to be specified later.

The representation consists in that to each element g of the group G we make correspond a unitary operator T_g in the space $\mathfrak{H}_m$, given by the formula

$$T_g f(h) = f(h\bar{g})\, a(h, g). \tag{6.2}$$

Here $a(h, g)$ is a fixed function giving the representation, which we shall specify later. $h\bar{g}$ denotes, as usual, that element h_1 of H_m into which the element h is mapped by means of the element g, that is to say $hg = s_1 h_1$, where $s_1 \in S_m$.

The condition $T_{g_1} T_{g_2} = T_{g_1 g_2}$ leads us, as in [1], to a functional equation for $a(h, g)$:

$$\alpha(h, g_1 g_2) = \alpha(h\bar{g}_1, g_2)\, \alpha(h, g_1). \tag{6.3}$$

We put

$$a(e, g) = a(g), \tag{6.4}$$

where e is the unit element of the group G. Putting in (6.3) $h = e$, $g_1 = h$, $g_2 = g$, we get:

$$a(hg) = a(h, g)\, a(h),$$

i.e.,

$$a(h, g) = \frac{a(hg)}{a(h)}. \tag{6.5}$$

Putting in (6.3) $h = e$, $g_1 = s \in S_m$, $g_2 = g$, and recalling that $e\bar{s} = e$, we get

$$a(sg) = a(s)\, a(g). \tag{6.6}$$

We proceed to find the condition which the function $a(h, g)$ must satisfy in order that the operators of the representation T_g should be unitary, i.e., so that for an arbitrary function $f(h)$ from $\mathfrak{H}_m$ there should hold the equality

$$(T_g f, T_g f) = (f, f). \tag{6.7}$$

In view of (6.1) and (6.2) this equation may be re-written in the form

$$\int |\alpha(h, g)|^2 |f(h\bar{g})|^2 \omega(h)\, d\mu_r(h) = \int |f(h_1)|^2 \omega(h_1)\, d\mu_r(h_1).$$

In the second integral we make the change of variable $h_1 = h\bar{g}$. In view of (3.22) we have:

$$\int |f(h_1)|^2 \omega(h_1)\, d\mu_r(h_1) = \int |f(h\bar{g})|^2 \omega(h\bar{g})\, \beta^{-1}(s_1)\, d\mu_r(h),$$

where the element $s_1 \in S_m$ is determined by the equation

$$hg = s_1 h_1, \tag{6.8}$$

and the function $\beta(s)$ is determined by (3.10) and (3.3).

Thus

$$\int |f(h\bar{g})|^2 |\alpha(h, g)|^2 \omega(h)\, d\mu_r(h) = \int |f(h\bar{g})|^2 \omega(h\bar{g})\, \beta^{-1}(s_1)\, d\mu_r(h).$$

Hence we get, in view of the arbitrary character of $f(h)$,

$$|\alpha(h, g)|^2 \omega(h) = \omega(h\bar{g})\, \beta^{-1}(s_1),$$

or

$$|\alpha(h, g)| = \beta^{-\frac{1}{2}}(s_1) \sqrt{\frac{\omega(h\bar{g})}{\omega(h)}}. \tag{6.9}$$

The condition (6.9) is the desired condition for the operators T_g to be unitary. Putting $h = e$ in (6.9) and assuming that

$$g = sh, \tag{6.10}$$

we get

$$|\alpha(g)| = |\alpha(sh)| = \beta^{-\frac{1}{2}}(s) \sqrt{\frac{\omega(h)}{\omega(e)}}. \tag{6.11}$$

In view of (6.6) this equation may be re-written in the form

$$|\alpha(s)|\, |\alpha(h)| = \beta^{-\frac{1}{2}}(s) \sqrt{\frac{\omega(h)}{\omega(e)}}. \tag{6.12}$$

Hence

$$|\alpha(h)| = \sqrt{\frac{\omega(h)}{\omega(e)}} \tag{6.13}$$

and

$$|\alpha(s)| = \beta^{-\frac{1}{2}}(s). \tag{6.14}$$

We put

$$\alpha(s) = \beta^{-\frac{1}{2}}(s)\, \chi(s). \tag{6.15}$$

By (6.14) we have

$$|\chi(s)| = 1. \tag{6.16}$$

Furthermore, since

$$\alpha(s_1 s_2) = \alpha(s_1)\,\alpha(s_2) \text{ and } \beta(s_1 s_2) = \beta(s_1)\,\beta(s_2) \quad (s_1, s_2 \in S_m),$$

we have

$$\chi(s_1 s_2) = \chi(s_1)\,\chi(s_2) \quad (s_1, s_2 \in S_m). \tag{6.17}$$

From (6.16) and (6.17) it follows that $\chi(s)$ is a character of the group S_m. Let us represent the element s of the group S_m in the form $s = \zeta c u$, where $\zeta \in Z_m$, $c \in C_m$ and $u \in U_m$ (see §3). Then

$$\chi(s) = \chi(\zeta)\,\chi(cu).$$

But $\chi(\zeta) = 1$ for all elements ζ of Z_m. Thus

$$\chi(\zeta c u) = \chi(cu) = \chi(c)\,\chi(u). \tag{6.18}$$

We now find expressions for the characters $\chi(c)$ and $\chi(u)$. Elements c of the group C_m have the form

$$c = [c_1, c_2, \cdots, c_{m+r}], \tag{6.19}$$

where the c_p for $p \leq m$ are second-order matrices of the form

$$c_p = \begin{pmatrix} \sqrt{\lambda_p} & 0 \\ 0 & \sqrt{\lambda_p} \end{pmatrix}, \quad \lambda_p > 0, \tag{6.20}$$

and $c_p = \lambda_p$ for $p = m + 1, \cdots, m + r$. Hence it follows that for $m \neq \frac{n}{2}$ the group C_m is isomorphic to the direct product of m multiplicative groups of positive real numbers and $r - 1$ multiplicative groups of all real numbers. For $m = \frac{n}{2}$ the group C_m is isomorphic to the direct product of multiplicative groups of positive real numbers. Consequently the characters on the group C_m have the form

$$\chi(c) = |\lambda_1|^{i\rho_1}\,|\lambda_2|^{i\rho_2} \ldots |\lambda_{m+\tau-1}|^{i\rho_{m+\tau-1}} \left(\frac{\lambda_{m+1}}{|\lambda_{m+1}|}\right)^{\epsilon_{m+1}} \cdots \left(\frac{\lambda_{m+\tau-1}}{|\lambda_{m+\tau-1}|}\right)^{\epsilon_{m+\tau-1}}. \tag{6.21}$$

Here $\rho_1, \rho_2, \cdots, \rho_{m+r-1}$ are arbitrary real numbers, the numbers $\epsilon_p (p = m + 1, \cdots, m + r - 1)$ equal 0 or 1 and the factor $(\lambda_p / |\lambda_p|)^{\epsilon_p}$ is accordingly equal to 1 or sign λ_p.

Elements u of the group U_m have the form:

$$u = [u_1, u_2, \cdots u_m, 1, \cdots, 1], \tag{6.22}$$

where the u_p are orthogonal second-order matrices:

$$u_p = \begin{pmatrix} u_1^{(p)} & -u_2^{(p)} \\ u_2^{(p)} & u_1^{(p)} \end{pmatrix}, \quad (u_1^{(p)})^2 + (u_2^{(p)})^2 = 1. \tag{6.23}$$

From this it follows that the group U_m is isomorphic to the direct product of

m groups of orthogonal second-order matrices. Consequently the characters on the group U_m have the form:

$$\chi(u) = (u_1^{(1)} + iu_2^{(1)})^{n_1} (u_1^{(2)} + iu_2^{(2)})^{n_2} \dots (u_1^{(m)} + iu_2^{(m)})^{n_m}, \qquad (6.24)$$

where $n_1, n_2, \cdots, n_m$ are integers.

Let us return to the expression for the function $a(h, g)$. We put

$$hg = s_1 h_1. \qquad (6.8)$$

Then, by (6.5) and (6.6), we have

$$\alpha(h, g) = \frac{\alpha(s_1 h_1)}{\alpha(h)} = \frac{\alpha(h_1)}{\alpha(h)} \alpha(s_1),$$

whence, by (6.15),

$$\alpha(h, g) = \frac{\alpha(h_1)}{\alpha(h)} \beta^{-\frac{1}{2}}(s_1) \chi(s_1). \qquad (6.25)$$

We find an analytic expression for the function $a(h, g)$ in terms of the parameters of the manifold Z_m.

We represent the element h in the form

$$h = \dot{d} x, \qquad (6.26)$$

where $\dot{d} \in \dot{D}_m$, $x \in X_m$. Almost every element xg can be represented, and moreover uniquely, in the form

$$xg = \hat{k} x_1, \qquad (6.27)$$

where $\hat{k} \in \hat{K}_m$, $x_1 \in X_m$.

The elements $\hat{k}$ and x_1 in the formula (6.27) constitute fractional-rational functions of the elements of the matrix x. Expressions for them may be found from the formulae of §2, sub-section 1, where in place of the matrix g it is necessary to take the matrix $g_1 = xg$.

Let

$$\hat{k} = \| k_{pq} \|. \qquad (6.28)$$

We put

$$k_{pp} = \begin{pmatrix} \alpha_p & \beta_p \\ \gamma_p & \delta_p \end{pmatrix} \quad (p = 1, \dots, m) \qquad (6.29)$$

and

$$\det k_{pp} = \Lambda_p \quad (p = 1, \cdots, m + r). \qquad (6.30)$$

The quantities a_p, β_p, γ_p, δ_p and Λ_p are expressible in terms of the elements of the matrix $g_1 = xg$ according to the formulae of §2, sub-section 1.

We denote by d the matrix

$$d = [d_1, d_2, \cdots, d_{m+r}], \text{ where } d_p = k_{pp}. \qquad (6.31)$$

Then $\hat{k} = \zeta_1 d$, where $\zeta_1 = \hat{k} d^{-1} \in Z_m$.

We represent further the matrix s_1 of (6.8) in the form

$$s_1 = \zeta c u \quad (\zeta \in Z_m,\ c \in C_m,\ u \in U_m), \tag{6.32}$$

where c has the form (6.19)-(6.20), and u has the form (6.22)-(6.23).

Finally, we represent the element $h_1 = h\bar{g}$ in the form

$$h_1 = \dot{d}'x' \quad (\dot{d}' \in \dot{D}_m,\ x' \in X_m). \tag{6.33}$$

From (6.8), (6.26) and (6.27) we obtain:

$$\dot{d}\hat{k}x_1 = s_1 h_1,$$

whence

$$\dot{d}\,\zeta_1 d x_1 = \zeta c u \dot{d}'x'$$

or

$$\zeta_2 \dot{d}\, d x_1 = \zeta c u \dot{d}'x', \tag{6.34}$$

where $\zeta_2 = \dot{d}\,\zeta_1 \dot{d}^{-1} \in Z_m$. By the uniqueness of the decomposition in the form $g = \zeta d x$, where $\zeta \in Z_m$, $d \in D_m$, $x \in X_m$, we find from (6.34):

$$x' = x_1 \tag{6.35}$$

and

$$\dot{d}d = c u \dot{d}'. \tag{6.36}$$

From the equation (6.36) it follows that

$$|\det c_p| = |\det d_p| = |\Lambda_p| \quad \text{for} \quad p = 1, \cdots, m \tag{6.37}$$

and

$$c_p = d_p = \Lambda_p \quad \text{for} \quad p = m+1, \cdots, m+\tau-1. \tag{6.37a}$$

In virtue of (6.21) we derive accordingly

$$\chi(c) = |\Lambda_1|^{i\rho_1} |\Lambda_2|^{i\rho_2} \ldots |\Lambda_{m+\tau-1}|^{i\rho_{m+\tau-1}} \left(\frac{\Lambda_{m+1}}{|\Lambda_{m+1}|}\right)^{\varepsilon_{m+1}} \cdots \left(\frac{\Lambda_{m+\tau-1}}{|\Lambda_{m+\tau-1}|}\right)^{\varepsilon_{m+\tau-1}} \tag{6.38}$$

Let us consider an element z_0 of the manifold Z_m [see formulae (4.8) and (4.9)]. To the element $h = \dot{d}x$ there corresponds in the manifold Z_m the element $\dot{z}x$, and to the element $h = \dot{d}'x_1$ an element $\dot{z}'x_1$, where

$$\dot{z} = z_0 \bar{\dot{d}},\ \dot{z}' = z_0 \bar{\dot{d}}' \quad \text{are elements of} \quad \dot{Z}_m. \tag{6.39}$$

We put

$$\dot{z} = [v_1, v_2, \ldots, v_m, 1, \ldots, 1], \quad \dot{z}' = [v_1', v_2', \ldots, v_m', 1, \ldots, 1], \tag{6.40}$$

where

$$v_p = \begin{pmatrix} 1 & 0 \\ z_p & 1 \end{pmatrix}, \quad v_p' = \begin{pmatrix} 1 & 0 \\ z_p' & 1 \end{pmatrix}, \quad p = 1, \ldots, m, \tag{6.41}$$

and let

$$\dot{d} = [\dot{d}_1, \dot{d}_2, \ldots, \dot{d}_{m+\tau}], \quad \dot{d}' = [\dot{d}_1', \dot{d}_2', \ldots, \dot{d}_{m+\tau}']. \tag{6.42}$$

The formulae (6.39) then mean that

$$v_p = v_0 \overline{\dot{d}_p}, \quad v'_p = v_0 \overline{\dot{d}'_p},$$

i.e.,

$$v_0 \dot{d}_p = k_p v_p, \quad v_0 \dot{d}'_p = k'_p v'_p \quad .(p = 1, \ldots, m), \tag{6.43}$$

where $v_0 = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$ and k_p, k'_p are triangular matrices of the form

$$k_p = \begin{pmatrix} k_{11}^{(p)} & k_{12}^{(p)} \\ 0 & k_{22}^{(p)} \end{pmatrix}, \quad k'_p = \begin{pmatrix} k_{11}^{\prime(p)} & k_{12}^{\prime(p)} \\ 0 & k_{22}^{\prime(p)} \end{pmatrix}. \tag{6.44}$$

A straightforward calculation gives us:

$$k_{22}^{(p)} = |\operatorname{Im} z_p|^{-\frac{1}{2}}, \quad k_{22}^{\prime(p)} = |\operatorname{Im} z'_p|^{-\frac{1}{2}}. \tag{6.45}$$

The formula (6.36) means that

$$\dot{d}_p d_p = c_p u_p \dot{d}'_p \quad (p = 1, \ldots, m). \tag{6.46}$$

From this it follows that

$$v_0 \overline{\dot{d}_p d_p} = v_0 \overline{c_p u_p \dot{d}'_p}.$$

In virtue of the fact that $v_0 \overline{c_p u_p} = v_0$, and by (6.43), we have:

$$v_p \bar{d}_p = v'_p. \tag{6.47}$$

The formula (6.47) gives us an expression for z'_p in terms of z_p and the elements α_p, β_p, γ_p, δ_p of the matrix d_p:

$$z'_p = \frac{\alpha_p z_p + \gamma_p}{\beta_p z_p + \delta_p}. \tag{6.48}$$

On the other hand, this formula means that

$$v_p d_p = \tilde{k}_p v'_p, \tag{6.49}$$

where $\tilde{k}_p$ is a triangular matrix of the form

$$\tilde{k}_p = \begin{pmatrix} \tilde{k}_{11}^{(p)} & \tilde{k}_{12}^{(p)} \\ 0 & \tilde{k}_{22}^{(p)} \end{pmatrix}. \tag{6.50}$$

Direct calculation shows that

$$\tilde{k}_{22}^{(p)} = \beta_p z_p + \delta_p. \tag{6.51}$$

Multiplying the equation (6.46) on the left by v_0 and using the commutativity of v_0 with c we obtain:

$$v_0 \dot{d}_p d_p = c_p v_0 u_p \dot{d}'_p$$

or

$$v_0 \dot{d}_p d_p = c_p (v_0 u_p v_0^{-1}) (v_0 \dot{d}'_p),$$

whence, by (6.43),

$$k_p v_p d_p = c_p (v_0 u_p v_0^{-1}) k'_p v'_p.$$

Substituting here in place of the matrix $v_p d_p$ its expression from (6.49) and cancelling out v_p' we get:

$$k_p \tilde{k}_p = c_p (v_0 u_p v_0^{-1}) k_p'. \tag{6.52}$$

The matrix $v_0 u_p v_0^{-1}$ is a triangular matrix of the same form as the matrices k_p, k_p' and $\tilde{k}_p$. As is easily verified by a simple calculation,

$$(v_0 u_p v_0^{-1})_{22} = u_1^{(p)} - i u_2^{(p)}.$$

The equation (6.52) gives:

$$k_{22}^{(p)} \tilde{k}_{22}^{(p)} = \sqrt{\lambda_p}\, (v_0 u_p v_0^{-1})_{22} k_{22}'^{(p)}$$

or, according to the values of these elements which were found above,

$$|\operatorname{Im} z_p|^{-\frac{1}{2}} (\beta_p z_p + \delta_p) = \sqrt{|\Lambda_p|}\, (u_1^{(p)} - i u_2^{(p)}) |\operatorname{Im} z_p'|^{-\frac{1}{2}}.$$

Hence we get

$$u_1^{(p)} - i u_2^{(p)} = \left| \frac{\operatorname{Im} z_p}{\operatorname{Im} z_p'} \right|^{-\frac{1}{2}} |\Lambda_p|^{-\frac{1}{2}} (\beta_p z_p + \delta_p). \tag{6.53}$$

The formula (6.24) for the character $\chi(u)$ thus assumes the form:

$$\chi(u) = \prod_{p=1}^{m} \left| \frac{\operatorname{Im} z_p}{\operatorname{Im} z_p'} \right|^{n_p/2} \prod_{p=1}^{m} (\beta_p z_p + \delta_p)^{-n_p} |\Lambda_p|^{n_p/2}. \tag{6.54}$$

Using (6.25) and taking into account that

$$\chi(s_1) = \chi(c)\, \chi(u),$$

where $\chi(c)$ and $\chi(u)$ are expressed by (6.38) and (6.54), we find the following expression for the function $a(h, g) = a(\dot{z}x, g)$:

$$\alpha(\dot{z}x, g) = \frac{\alpha(\dot{z}' x_1)}{\alpha(\dot{z}x)} \prod_{p=1}^{m} \left| \frac{\operatorname{Im} z_p}{\operatorname{Im} z_p'} \right|^{n_p/3} \cdot \prod_{p=1}^{m} (\beta_p z_p + \delta_p)^{-n_p} |\Lambda_p|^{n_p/2}.$$

$$\prod_{p=1}^{m+\tau-1} |\Lambda_p|^{i\rho_p} \cdot \prod_{p=m+1}^{m+\tau-1} \left(\frac{\Lambda_p}{|\Lambda_p|} \right)^{\varepsilon_p} \beta^{-\frac{1}{2}}(s_1), \tag{6.55}$$

where, by (3.10),

$$\beta(s_1) = |\Lambda_2|^{r_1 + r_2} |\Lambda_3|^{r_1 + 2r_2 + r_3} \dots |\Lambda_{m+\tau}|^{r_1 + 2r_2 + \dots + 2r_{m+\tau-1} + r_{m+\tau}}. \tag{6.56}$$

According to §5, the function $a(\dot{z}x, g)$ must be an analytic function of z_1, $z_2, \cdots, z_m$ so that the expression for $a(\dot{z}x, g)$ is not to contain the factor

$$\prod_{p=1}^{m} \left| \frac{\operatorname{Im} z_p}{\operatorname{Im} z_p'} \right|^{n_p/2}.$$

Hence we put

$$\alpha(\dot{z}x) = \prod_{p=1}^{m} |\operatorname{Im} z_p|^{n_p/2}. \tag{6.57}$$

The formula for $a(\dot{z}x, g)$ takes then the following final form:

$$a(\dot{z}x, g) = \prod_{p=1}^{m} (\beta_p z_p + \delta_p)^{-n_p} |\Lambda_p|^{n_p/2} \prod_{p=1}^{m+\tau-1} |\Lambda_p|^{i\rho_p} \prod_{p=m+1}^{m+\tau-1} \left(\frac{\Lambda_p}{|\Lambda_p|}\right)^{\varepsilon_p} \beta^{-\frac{1}{2}}(s_1). \tag{6.58}$$

It remains to determine the *scalar product* in the space $\mathfrak{H}_m$.

In view of (6.13) and (6.57) we have:

$$\frac{\omega(h)}{\omega(\varepsilon)} = \prod_{p=1}^{m} |\operatorname{Im} z_p|^{n_p}$$

or

$$\omega(h) = c \prod_{p=1}^{m} |\operatorname{Im} z_p|^{n_p}, \tag{6.59}$$

and the formula for the scalar product takes the form:

$$(f_1, f_2) = c \int f_1(h) \overline{f_2(h)} \prod_{p=1}^{m} |\operatorname{Im} z_p|^{n_p} d\mu_r(h). \tag{6.60}$$

By (3.15) and (1.16), the expression for the right-invariant measure in the parameters of the manifold Z_m has the form

$$d\mu_r(h) = \frac{1}{2^m} \prod_{p=1}^{m} (\operatorname{Im} z_p)^{-2} d\dot{z}\, d\mu(x),$$

where $d\dot{z} = dx_1 dy_1 \cdots dx_m dy_m$ $(z_p = x_p + iy_p)$. Hence, putting

$$f(h) = f(z_1, z_2, \cdots, z_m, x)$$

and making a suitable choice of a multiplier, we obtain the following final formula for the scalar product in the space $\mathfrak{H}_m$:

$$(f_1, f_2) = c \int f_1(z_1, \ldots, z_m, x) \overline{f_2(z_1, \ldots, z_m, x)} \prod_{p=1}^{m} |\operatorname{Im} z_p|^{n_p - 2} d\dot{z}\, d\mu(x), \tag{6.61}$$

where

$$z_p = x_p + iy_p \quad (p = 1, \cdots, m), \qquad d\dot{z} = dx_1 dy_1 \cdots dx_m dy_m. \tag{6.62}$$

As is well-known, in analysis instead of the power-function x^λ $(x \geq 0)$ it is more natural to consider the function $x^\lambda / \Gamma(\lambda + 1)$. For $\lambda = -1$ this function changes to the delta-function $\delta(x)$. Accordingly in (6.61) we take

$$\frac{1}{c} = \prod_{p=1}^{m} \Gamma(n_p - 1).$$

The formula for the scalar product in the space $\mathfrak{H}$ takes then the form:

$$(f_1, f_2) = \int f_1(z_1, \ldots, z_m, x)\,\overline{f_2(z, \ldots, z_m, x)} \prod_{p=1}^{m} \frac{|\operatorname{Im} z_p|^{n_p-2}}{\Gamma(n_p-1)}\, d\dot{z}\, d\mu(x). \tag{6.61a}$$

From convergence considerations for the integral we conclude that all the numbers n_p must be positive. In the case when one or several of the numbers n_p are equal to unity the corresponding factors $|\operatorname{Im} z_p|^{n_p-2}/\Gamma(n_p-1)$ in (6.61a) are to be regarded as delta-functions or, what is the same thing, the integral in (6.61) should be regarded as a limit as $n_p \to 1$.

All the above arguments are valid also for $m = \frac{n}{2}$, and lead us to the series $d^{+}_{n/2}$ and $d^{-}_{n/2}$, realisable on functions over the manifolds $Z^{+}_{n/2}$ and $Z^{-}_{n/2}$.

We formulate the basic result:

The principal nondegenerate unitary representations of the real unimodular n-th order group G fall, if n is odd, into $(n+1)/2$ *series* $d_0, d_1, d_2, \cdots, d_{(n-1)/2}$, *and if n is even into* $\frac{n}{2}+2$ *series* $d_0, d_1, \cdots, d_{n/2-1}, d^{+}_{n/2}, d^{-}_{n/2}$.

The representations of each series d_m *for* $m \neq \frac{n}{2}$ *are determined by a system of m positive integers* $n_1, n_2, \cdots, n_m$, *a system of* $m+r-1$ *real numbers* $\rho_1, \rho_2, \cdots, \rho_{m+r-1}$ $(2m+r=n)$ *and* $r-1$ *indices* $\epsilon_{m+1}, \epsilon_{m+2}, \cdots, \epsilon_{m+r-1}$, *which may take the value 0 or 1.*

The representation which corresponds to a given system of numbers n_p, ρ_q, ϵ_s, $(p=1,\cdots,m;\ q=1,\cdots,m+r-1;\ s=m+1,\cdots,m+r-1)$, *is formed in a space of functions* $f(\dot{z}x) = f(z_1, z_2, \cdots, z_m, x)$, *defined on the transitive manifold* Z_m, *which in respect of each variable* z_p *(for fixed values of the remaining variables) are defined and analytic in the upper half-plane* $\operatorname{Im} z_p > 0$ *and the lower half-plane* $\operatorname{Im} z_p < 0$ *separately.*

The scalar product in this space is given by (6.61):

$$(f_1, f_2) = \int f_1(z_1, \ldots, z_m, x)\,\overline{f_2(z_1, \ldots, z_m, x)}\cdot$$

$$\cdot \prod_{p=1}^{m} \frac{|\operatorname{Im} z_p|^{n_p-2}}{\Gamma(n_p-1)}\, d\mu(x)\cdot dx_1\, dy_1 \ldots dx_m\, dy_m,$$

where

$$z_p = x_p + iy_p \quad (p=1,\cdots,m).$$

The operators of the representation T_g *are given by the formula*

$$T_g f(z_1, z_2, \ldots, z_m, x) = f(z'_1, z'_2, \ldots, z'_m, x_1)\,\alpha(\dot{z}x, g), \tag{6.63}$$

where

$$z'_p = \frac{\alpha_p z_p + \gamma_p}{\beta_p z_p + \delta_p} \quad (p = 1, \ldots, m),$$

$$\alpha(\dot{z}x, g) = \prod_{p=1}^{m} (\beta_p z_p + \delta_p)^{-n_p} |\Lambda_p|^{n_p/2} \cdot \prod_{p=1}^{m+\tau-1} |\Lambda_p|^{i\rho_p} \prod_{p=m+1}^{m+\tau-1} \left(\frac{\Lambda_p}{|\Lambda_p|}\right)^{\varepsilon_p} \cdot$$

$$\cdot \{|\Lambda_2|^{r_1+r_2} |\Lambda_3|^{r_1+2r_2+r_3} \ldots |\Lambda_{m+\tau}|^{r_1+2r_2+\cdots+2r_{m+\tau-1}+r_{m+\tau}}\}^{-\frac{1}{2}} \tag{6.64}$$

$$(r_p = 2 \text{ for } p \leq m \text{ and } 1 \text{ for } p > m),$$

or, what is the same thing,

$$\alpha(\dot{z}x, g) = \prod_{p=1}^{m} \left[\frac{\beta_p z_p + \delta_p}{\sqrt{|\alpha_p\delta_p - \beta_p\gamma_p|}}\right]^{-n_p} \cdot \prod_{p=1}^{m+\tau-1} |\Lambda_p|^{i\rho_p} \cdot \prod_{p=m+1}^{m+\tau-1} \left(\frac{\Lambda_p}{|\Lambda_p|}\right)^{\varepsilon_p} \cdot$$

$$\cdot |\Lambda_1|^{n-\frac{3}{2}} |\Lambda_2|^{n-2-\frac{3}{2}} \ldots |\Lambda_m|^{\tau+\frac{1}{2}} |\Lambda_{m+1}|^{\tau-1} |\Lambda_{m+2}|^{\tau-2} \ldots |\Lambda_{m+\tau-1}|.$$

Here the elements of the matrix x_1, *the quantities* $\alpha_p, \beta_p, \gamma_p, \delta_p$, $(p = 1, \cdots, m)$, *and also* Λ_p $(p = m+1, \cdots, m+\tau)$, *are fractional-rational functions of the elements of the matrix* x *and can be found from the formulae given in* §2, *sub-section* 1.

For $m = \frac{n}{2}$ *the representations of the series* d_m^+ *and* d_m^- *are fixed by a system of* m *positive integers* $n_1, n_2, \cdots, n_m$ *and* $m - 1$ *real numbers* $\rho_1, \rho_2, \cdots, \rho_{m-1}$.

A representation of either of these series, corresponding to the system of numbers n_p *and* ρ_q $(p = 1, \cdots, m;\ q = 1, \cdots, m-1)$, *is formed in a space of functions*

$$f(\dot{z}x) = f(z_1, z_2, \cdots, z_m, x),$$

defined on the transitive manifold Z_m^+ *or, as the case may be,* Z_m^-. *These functions are in respect of each variable* z_p *(for fixed values of the remaining variables) defined and analytic in the upper or the lower half-plane, the choice of which being fixed by the sign of the expression*

$$\operatorname{Im} z_1 \cdots \operatorname{Im} z_{p-1} \operatorname{Im} z_{p+1} \cdots \operatorname{Im} z_m.$$

The scalar product has the same form as for $m \neq \frac{n}{2}$.

The operators of the representation T_g *are given, in a similar manner to the case* $m \neq \frac{n}{2}$, *by the formula*

$$T_g f(z_1, z_2, \ldots, z_m, x) = f(z'_1, z'_2, \ldots, z'_m, x_1)\, \alpha(\dot{z}x, g), \tag{6.65}$$

where

$$z'_p = \frac{\alpha_p z_p + \gamma_p}{\beta_p z_p + \delta_p} \quad (p = 1, \ldots, m),$$

$$\alpha(\dot{z}x, g) =$$

$$= \prod_{p=1}^{m} (\beta_p z_p + \delta_p)^{-n_p} |\Lambda_p|^{n_p/2} \prod_{p=1}^{m-1} |\Lambda_p|^{i\rho_p} \cdot |\Lambda_2|^{-2} |\Lambda_3|^{-4} \ldots |\Lambda_m|^{-(n-2)}, \quad (6.66)$$

or, what is the same thing,

$$\alpha(\dot{z}x, g) =$$

$$= \prod_{p=1}^{m} \left[\frac{\beta_p z_p + \delta_p}{\sqrt{|\alpha_p \delta_p - \beta_p \gamma_p|}} \right]^{-n_p} \cdot \prod_{p=1}^{m-1} |\Lambda_p|^{i\rho_p} \cdot |\Lambda_1|^{n-2} |\Lambda_2|^{n-4} \cdots |\Lambda_{m-1}|^{2}.$$

§7. Irreducibility of representations of the principal nondegenerate series

1. In this section we shall consider, side by side with the group G_n, also the group $\widetilde{G}_n$ of all real n-th order matrices with determinant ± 1.

It is easy to see that for $r > 0$ the formulae (6.63), (6.64) give, as g runs through the group $\widetilde{G}_n$, a unitary representation of this group, realised in the same function-space as the corresponding representation of the group G_n.

In the case $r = 0$ the manifolds $Z^+_{n/2}$ and $Z^-_{n/2}$ are not transitive with respect to mappings by elements of the group $\widetilde{G}_n$. Howerer, their sum

$$Z_{n/2} = Z^+_{n/2} + Z^-_{n/2}$$

is a transitive manifold. We shall consider a representation of the group $\widetilde{G}_n$, which is realisable in the space of functions

$$f(\dot{z}x) = f(z_1, \cdots, z_m, x),$$

defined on this transitive manifold $Z_{n/2}$, which are analytic in each of the variables z_p, for fixed values of the remaining variables, separately in the upper and lower half-plane. We shall fix the scalar product in this space by the formula (6.61):

$$(f_1, f_2) = \int f_1(z_1, \ldots, z_m, x) \overline{f_2(z_1, \ldots, z_m, x)} \cdot$$

$$\cdot \prod_{p=1}^{m} \frac{|\operatorname{Im} z_p|^{n_p - 2}}{\Gamma(n_p - 1)} \, d\mu(x) \, dx_1 \, dy_1 \ldots dx_m \, dy_m,$$

where the integral is taken over the whole manifold $Z_{n/2}$. We shall give the operators of the representation T_g by the same formulae (6.65), (6.66) as for the group G_n. It is easy to see that the representation we have described is unitary.

We thus obtain the principal nondegenerate series of unitary representations of the group $\widetilde{G}_n$, which in what follows we shall denote by d_m $(m = 0, 1, \cdots, [\frac{n}{2}])$ as in the case of the group G_n.

The basic result of this section is the following

Theorem 1. *All the representations of the principal nondegenerate series of* *the group* $\widetilde{G}_n$ *of real matrices with determinant* ± 1 *are irreducible.*

For $m = \frac{n}{2}$ the representation of the group G_n of real matrices with determinant $+1$, given by a representation of the group $\widetilde{G}_n$ from the series $d_{n/2}$, splits up into representations from the series $d^+_{n/2}$ and $d^-_{n/2}$ with the same numbers n_p and ρ_q as the original representation of the group $\widetilde{G}_n$. In view of this Theorem 1 there follows immediately the irreducibility of representations of the group G_n from the series $d^+_{n/2}$ and $d^-_{n/2}$ (for even n). Apart from this, in as much as a group $\widetilde{G}_n$ is for odd n the direct product of the group G_n and the second-order cyclic group, from Theorem 1 there follows also the irreducibility of representations of the principal nondegenerate series of the group G_n for odd n.

2. In place of the group $\widetilde{G}_n$ we introduce a certain subgroup of it, and show that even representations of this subgroup are irreducible. At a first glance it might appear that the proof of the irreducibility of the representations of a subgroup $\widetilde{G}_n$ would not in any case be simpler than the proof of the irreducibility of the representations of the whole group $\widetilde{G}_n$. However, it turns out in fact to be possible, as we shall see below, to carry out the proof of the irreducibility of the representations by induction with respect to n if we consider instead of the group $\widetilde{G}_n$ a suitable subgroup of it.

As such a subgroup of the group $\widetilde{G}_n$ we take, if n is even, the subgroup $\widetilde{\mathfrak{A}}_n$ of all matrices g which satisfy the condition

$$g_{np} = 0 \quad \text{for} \quad p = 1, 2, \cdots, n-1, \tag{*}$$

and also the condition

$$g_{nn} > 0. \tag{**}$$

If n is odd we choose the subgroup $\mathfrak{A}_n$ of all matrices with determinant ± 1 satisfying only the condition (*).

There holds the following

Lemma 1. *If* T_g *is any of the representations of the principal nondegenerate series of the group* $\widetilde{G}_n$, *then if* n *is even* T_g *is an irreducible representation of the subgroup* $\widetilde{\mathfrak{A}}_n$, *and if* n *is odd an irreducible representation of the subgroup* $\mathfrak{A}_n$.

It is obvious that Theorem 1 is an immediate consequence of this lemma.

3. We shall carry out the proof of Lemma 1 by induction. We assume that the assertion of the lemma has been already proved for the group of $(n-1)$-th order, matrices, and prove it for the group of n-th order matrices.

We consider separately various possible cases, according as to whether n is an even or an odd number, and again according to the type of the series of representations in question*.

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* The proof given below for cases I and II is in essence the same as the proof of the irreducibility of the representations of a principal nondegenerate series of the complex unimodular group (see [1]).

Case I. n an odd number. A representation of the group G_n from the series $d_{m,r}$ $(2m+r=n)$ is determined by m positive integers $n_1, n_2, \cdots, n_m$, $m+r-1$ real numbers $\rho_1, \rho_2, \cdots, \rho_{m+r-1}$, and also, if $r>1$, indices $\epsilon_{m+1}, \cdots, \epsilon_{m+r-1}$. The representation in this case realised in the space $\mathfrak{H}$ of functions

$$f(z) = f(\dot{z}x) = f(z_1, \cdots, z_m, x),$$

defined on the manifold $Z_{m,r}$ and analytic in each of the variables $z_1, z_2, \cdots, z_m$. The norm in the space $\mathfrak{H}$ is given by the formula

$$\|f\|^2 = \int |f(z_1, \ldots, z_m, x)|^2 \prod_{p=1}^{m} \frac{|\operatorname{Im} z_p|^{n_p-2}}{\Gamma(n_r-1)} \cdot$$

$$\cdot d\mu(x)\, dx_1\, dy_1 \ldots dx_m\, dy_m \quad (z_p = x_p + iy_p). \tag{7.1}$$

The operators of the representation T_g are given by the formula

$$T_g f(z) = f(z\bar{g}) \left[\prod_{p=1}^{m} \frac{\beta_p z_p + \delta_p}{\sqrt{|\alpha_p\delta_p - \beta_p\gamma_p|}} \right]^{-n_p} \prod_{p=1}^{m+\tau-1} |\Lambda_p|^{i\rho_p} \cdot$$

$$\cdot \prod_{p=m+1}^{m+\tau-1} \left(\frac{\Lambda_p}{|\Lambda_p|} \right)^{\epsilon_p} \cdot |\Lambda_1|^{n-\frac{3}{2}} |\Lambda_2|^{n-\frac{7}{2}} \cdots$$

$$\cdots |\Lambda_m|^{\tau+\frac{1}{2}} |\Lambda_{m+1}|^{\tau-1} |\Lambda_{m+2}|^{\tau-2} \cdots |\Lambda_{m+\tau-1}|. \tag{7.2}$$

We denote by z' an $(n-1)$-th order matrix whose elements are the elements z_{pq} of the matrix $z \in Z_{m,r}$ for which $p<n$, $q<n$. We remark that the elements z_{nq} of the bottom row of the matrix $z \in Z_{m,r}$ coincide with the corresponding elements x_{nq} of the matrix $x \in X_{m,r}$ which appears in the representation $z = \dot{z}x$ of the matrix z, where $\dot{z} \in Z_{m,r}$, $x \in X_{m,r}$. By what has been said, one can represent the function $f(z) \in \mathfrak{H}$ in the form

$$f(z) = f(z', x_{n1}, x_{n2}, \cdots, x_{n,n-1}), \tag{7.3}$$

where $z' \in Z_{m,r-1}$ and for fixed z' the function f is of summable square with respect to $x_{n1}, x_{n2}, \cdots, x_{n,n-1}$.

We effect a Fourier transformation, putting

$$\varphi(z', w_1, \ldots, w_{n-1}) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int f(z', x_{n1}, \ldots, x_{n,n-1}) \cdot$$

$$\cdot e^{-i(x_{n1}w_1 + \cdots + x_{n,n-1}w_{n-1})}\, dx_{n1} \ldots dx_{n,n-1}. \tag{7.4}$$

This transformation will be a unitary mapping of the space $\mathfrak{H}$ on to the space H of functions $\phi(z', w_1, \cdots, w_{n-1})$ with the norm

$$\|\varphi\|^2 = \int |\varphi(z', w_1, \ldots, w_{n-1})|^2 \cdot \prod_{p=1}^{m} |\operatorname{Im} z_p|^{n_p-2} dz'\, dw_1 \ldots dw_{n-1}.^{*} \tag{7.5}$$

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[*] For brevity we shall in future denote the expression $d\mu(x)dx_1dy_1 \cdots dx_m dy_m$ in (7.1) by dz; dz' will represent the corresponding expression for z'.

Hence all operators of the representation may be regarded as operators in H.

Let A be a bounded operator in H, which commutes with all operators T_g, $g \in \mathfrak{A}_n$. The assertion of the lemma will be proved, for the case in question, if we prove that such an operator is a multiple of the identity.

The group $\mathfrak{A}_n$ contains the subgroup $X_{m,\tau}$, and in particular it contains all elements $x^0 \in X_{m,\tau}$ defined by the conditions

$$x^0_{pq} = 0 \quad \text{for} \quad p < n. \tag{7.6}$$

In the space $\mathfrak{H}$ the operator T_{x^0} is the displacement operator:

$$T_{x^0} f(z) = f(zx^0). \tag{7.7}$$

On the other hand, it follows from the condition (7.6) that elements z' remain unchanged under the displacement $z \to zx^0$, and $x_{nq} \to x_{nq} + x^0_{nq}$, $q = 1, 2, \cdots, n-1$. Hence (7.7) may be re-written in the form

$$T_{x^0} f(z', x_{n1}, \ldots, x_{n,n-1}) = f(z', x_{n1} + x^0_{n1}, \ldots, x_{n,n-1} + x^0_{n,n-1}). \tag{7.8}$$

Hence, in virtue of (7.4), the operator T_{x^0} in the space H will have the form:

$$T_{x^0}\varphi(z', w_1, \ldots, w_{n-1}) = e^{i\left(x^0_{n1}w_1+\cdots+x^0_{n,n-1}w_{n-1}\right)}\varphi(z', w_1, \ldots, w_{n-1}). \tag{7.9}$$

By hypothesis, the operator A commutes with all the operators T_{x^0}, whatever may be the values of the numbers $x^0_{n1}, \cdots, x^0_{n,n-1}$, and consequently with all operators of multiplication by bounded functions $\omega(w_1, \cdots, w_{n-1})$. Hence the operator A has the form:

$$A\varphi(z', w_1, \ldots, w_{n-1}) = a(w_1, \ldots, w_{n-1})\varphi(z', w_1, \ldots, w_{n-1}), \tag{7.10}$$

where $a(w_1, \cdots, w_{n-1})$ is an operator in the space of functions $f(z')$ with the norm

$$\|f\|^2 = \int |f(z')|^2 \prod_{p=1}^{m} \frac{|\operatorname{Im} z_p|^{n_p-2}}{\Gamma(n_p-1)}\, dz', \tag{7.11}$$

defined for almost all $w_1, \cdots, w_{n-1}$ and uniformly bounded in these variables.

We consider in the group $\mathfrak{A}_n$ the subgroup of matrices which satisfy the additional condition

$$g_{nq} = 0 \quad \text{for} \quad q = 1, 2, \cdots, n-1. \tag{7.12}$$

We denote this subgroup by $\mathfrak{A}^0$. The operator A must commute with all operators T_g, $g \in \mathfrak{A}^0$. We proceed to find these operators T_g.

In accordance with (5.44) and (5.45) of [1] we have (cf. also the formulae cited in §2, sub-section 1 of the present article) the following expressions for elements of the matrix $\hat{z} = \bar{z}g$:

$$\hat{z}_{pq} = \frac{\begin{vmatrix} g'_{pq} & g'_{p,\,p+1} \cdots g'_{p,\,n-1} \\ \cdots & \cdots \\ g'_{n-1,\,q} & g'_{n-1,\,p+1} \cdots g'_{n-1,\,n-1} \end{vmatrix}}{\begin{vmatrix} g'_{pp} & g'_{p,\,p+1} \cdots g'_{p,\,n-1} \\ \cdots & \cdots \\ g'_{n-1,\,p} & g'_{n-1,\,p+1} \cdots g'_{n-1,\,n-1} \end{vmatrix}} \quad \text{for } q < p < n, \tag{7.13}$$

where g'_{pq} are elements of the matrix $g' = zg$,

$$\hat{x}_{nq} = \hat{z}_{nq} = \sum_{s=1}^{n-1} x_{ns} \frac{g_{sq}}{g_{nn}} \quad \text{for } q = 1, \cdots, n-1. \tag{7.14}$$

We denote by g^0 the $(n-1)$-th order matrix whose elements are

$$g^0_{pq} = g_{pq} \cdot c, \quad q,\, p < n, \tag{7.15}$$

where the real multiplier c is chosen so that the determinant of the matrix g^0 equals 1 or -1. It is obvious that the sign of $\det g^0$ coincide with the sign of the element g_{nn} of the matrix g. We shall assume that the sign of the number c, the choice of which is entirely at our disposal, also coincides with the sign of the number g_{nn}. We put further

$$b = g_{nn} \cdot c \;{}^{*}. \tag{7.16}$$

Then $b > 0$. From (7.13) and (7.14) it follows that under the mapping $z \to z\bar{g}$ the matrix z' goes into $z'\overline{g^0}$, and the x_{nq} go into $\hat{x}_{nq}$, where

$$b\hat{x}_{nq} = \sum_{s=1}^{n-1} g^0_{sq}\, x_{ns}. \tag{7.17}$$

Let us calculate $a(z, g)$. According to results obtained above [see (6.30), (2.3)], the expressions Λ_p which occur in the formula for $a(z, g)$ can be calculated from the formulae

$$\Lambda_p = \frac{\tilde{g}_{r_1+\cdots+r_{p-1}+1}}{\tilde{g}_{r_1+\cdots+r_p+1}}, \tag{7.18}$$

where $\tilde{g}_k$ denotes as usual the principal minor of the matrix $\tilde{g} = xg$,

$$\tilde{g}_k = \begin{vmatrix} \tilde{g}_{kk} \cdots \tilde{g}_{kn} \\ \cdots\cdots \\ \tilde{g}_{nk} \cdots \tilde{g}_{nn} \end{vmatrix}.$$

We recall also that $r_1 = r_2 = \cdots = r_m = 2$, $r_{m+1} = \cdots = r_{m+r} = 1$.

We represent a matrix $z' \in Z_{m,\,r-1}$ in the form

$$z' = \dot{z}'x', \quad \dot{z}' \in \dot{Z}_{m,\,r-1}, \quad x' \in X_{m,\,r-1}.$$

---

[*] Since $\det g = 1$, it follows from the equation (7.15) that $g_{nn} = \pm c^{n-1}$, and from (7.16) that $b = \pm c^n$.

The matrices $\dot{z}'$ and x' are obtained from the corresponding matrices $\dot{z}$ and x, occuring in the representation of the matrix $z\,(z = \dot{z}x)$, by striking out the bottom row and the last column. If we put $\tilde{g}' = x'g^0$, then the principal minors $\tilde{g}_p$ for $p<n$ of the matrix $\tilde{g} = xg$ will differ from the principal minors $\tilde{g}_p$ of the $(n-1)$-th order matrix $\tilde{g}'$ only by a constant factor, being the product of g_{nn} and a certain power of the number c. Hence the expressions

$$\Lambda'_p = \frac{\tilde{g}'_{r_1+\cdots+r_{p-1}+1}}{\tilde{g}'_{r_1+\cdots+r_p+1}}, \quad p = 1, 2, \ldots, m+\tau-1,$$

differ from the corresponding expressions Λ_p by a constant factor, being a certain power of the number c. Hence, since

$$g_{nn} = c^{1-n} \operatorname{sign} c \ \text{ and } \ \Lambda'_{m+\tau-1} = \frac{1}{\Lambda'_1 \ldots \Lambda'_{m+\tau-2}},$$

we have

$$\alpha\,(z, g) = \alpha_0\,(g_{nn})\,\alpha'\,(z', g^0), \tag{7.19}$$

where $\alpha_0\,(g_{nn})$ is a function of the form

$$\alpha_0\,(g_{nn}) = g^s_{nn}\,|\,g_{nn}\,|^{l+i\rho} \tag{7.20}$$

and

$$\alpha'\,(z', g^0) = \prod_{p=1}^{m}\left[\frac{\beta_p z_p + \delta_p}{\sqrt{|\,\alpha_p\delta_p - \beta_p\gamma_p\,|}}\right]^{-n_p} \prod_{p=2}^{m+\tau-2} |\,\Lambda'_p\,|^{i(\rho_p - \rho_{m+\tau-1})} \cdot$$
$$\cdot \prod_{p=1}^{m+\tau-2}\left(\frac{\Lambda'_p}{|\Lambda'_p|}\right)^{\varepsilon_p} |\,\Lambda'_1\,|^{(n-1)-3/2} \ldots |\,\Lambda'_m\,|^{(\tau-1)+1/2} \cdot$$
$$\cdot\ |\,\Lambda'_{m+1}\,|^{(\tau-1)-1} \ldots |\,\Lambda'_{m+\tau-2}\,| \cdot \left(\frac{\Lambda'_{m+\tau-1}}{|\,\Lambda'_{m+\tau-1}\,|}\right)^{\varepsilon_{m+\tau-1}}. \tag{7.21}$$

The function $\alpha'(z', g^0)$ differs for $\tau > 1$ from the function, giving a unitary representation of the group $\tilde{G}_{n-1}$ from the series $d_{m,\,\tau-1}$, solely by a factor

$$(\Lambda'_{m+\tau-1} / |\,\Lambda'_{m+\tau-1}\,|)^{\varepsilon_{m+\tau-1}}.$$

For $\tau = 1$ this function coincides with the function which gives a unitary representation of the group G_{n-1} from the series $d_{m,\,0}$.

Thus, for $g \in \mathfrak{A}^0$,

$$T_g f\,(z', x_{n1}, \ldots, x_{n,\,n-1}) = \alpha_0\,(g_{nn})\,\alpha'\,(z', g^0)\,f\,(z'\overline{g^0}, \hat{x}_{n1}, \ldots, \hat{x}_{n.\,n-1}) \tag{7.22}$$

or

$$T_g f\,(z', x_{n1}, \ldots, x_{n,\,n-1}) = \alpha_0\,(g_{nn})\,T_{g^0} f\,(z', \hat{x}_{n1}, \ldots, \hat{x}_{n,\,n-1}), \tag{7.23}$$

where $\hat{x}_{n1}, \cdots, \hat{x}_{n,\,n-1}$ are determined by (7.17) and T_{g^0} is a representation of the group $\widetilde{G}_{n-1}$, determined by the equation

$$T_{g^0} f(z') = \alpha'(z', g^0)\, f(z'\overline{g}^0). \tag{7.24}$$

Let us see how the operator T_g acts on the function $\phi(z', w_1, \cdots, w_{n-1})$. We have

$$T_g \varphi(z', w_1, \ldots, w_{n-1}) = \alpha_0(g_{nn})\, \alpha'(z', g^0) \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int f(z'\overline{g}^0, \hat{x}_{n1}, \ldots, \hat{x}_{n,n-1}) \cdot$$

$$\cdot\, e^{-i(x_{n1}w_1 + \cdots + x_{n,n-1}w_{n-1})}\, dx_{n1} \ldots dx_{n,n-1}.$$

Transferring to the variables of integration $\hat{x}_{n1}, \cdots, \hat{x}_{n,n-1}$ we get

$$T_g \varphi(z', w_1, \ldots, w_{n-1}) = \alpha_0(g_{nn})\, \alpha'(z', g^0)\, \Delta \cdot \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int f(z'\overline{g}^0, \hat{x}_{n1}, \ldots, \hat{x}_{n,n-1}) \cdot$$

$$\cdot\, e^{-i(\hat{x}_{n1}\hat{w}_1 + \cdots + \hat{x}_{n,n-1}\hat{w}_{n-1})}\, d\hat{x}_{n1} \ldots d\hat{x}_{n,n-1}, \tag{7.25}$$

that is to say

$$T_g \varphi(z', w_1, \ldots, w_{n-1}) = \alpha_1(g_{nn})\, \Delta \cdot T_{g^0} \varphi(z', \hat{w}_1, \ldots, \hat{w}_{n-1}). \tag{7.26}$$

Here Δ is the Jacobian of the transformation (7.17) of the variables x_{nq}, and $\hat{w}_q$ are given by

$$\frac{1}{b} \hat{w}_q = \sum_{s=1}^{n-1} \hat{g}_{sq} w_s, \tag{7.27}$$

where the matrix $\hat{g} = \| g_{sq} \|$ is the inverse of the adjoint matrix to g^0, i.e., $\hat{g} = (g^0)^{*-1}$.

The operator A must commute with the operator T_g. In view of the formulae obtained above this gives:

$$a(w_1, \ldots, w_{n-1}) \cdot \Delta \cdot \alpha_0(g_{nn})\, T_{g^0} \varphi(z', \hat{w}_1, \ldots, \hat{w}_{n-1}) =$$
$$= \Delta \cdot \alpha_0(g_{nn})\, T_{g^0} a(\hat{w}_1, \ldots, \hat{w}_{n-1})\, \varphi(z', \hat{w}_1, \ldots, \hat{w}_{n-1}),$$

and so

$$a(w_1, \ldots, w_{n-1})\, T_{g^0} = T_{g^0} a(\hat{w}_1, \ldots, \hat{w}_{n-1}).$$

Hence

$$a(\hat{w}_1, \ldots, \hat{w}_{n-1}) = T_{g^0}^{-1} a(w_1, \ldots, w_{n-1})\, T_{g^0} \tag{7.28}$$

for almost all $w_1, \cdots, w_{n-1}$.

In (7.15) we put $g^0 = e$ (e being the unit matrix). Then (7.27) takes the form

$$\frac{1}{b} \hat{w}_q = w_q, \quad b > 0,$$

and $T_{g^0} = 1$. Hence the relation (7.28) may be re-written in the form

$$a(bw_1, \ldots, bw_{n-1}) = a(w_1, \ldots, w_{n-1}), \tag{7.29}$$

where $b > 0$. In view of this, the function $a(w_1, \cdots, w_{n-1})$ is constant along rays which issue from the point $(0, 0, \cdots, 0)$. It may therefore be regarded as a function of $w \in \mathfrak{W}$, where $\mathfrak{W}$ is the $(n-2)$ – dimensional sphere $w_1^2 + \cdots + w_{n-1}^2 = 1$.

Since the transition $g^0 \to \hat{g} = (g^0)^{*-1}$ is an isomorphism, it follows that the equation (7.27) may be considered as a mapping of the manifold $\mathfrak{B}$ by means of the group $\tilde{G}_{n-1}$ of $(n-1)$-th order matrices with determinant ± 1.

It is obvious that $\mathfrak{B}$ is transitive, so that it may be regarded as a manifold of right cosets with respect to some subgroup of the group $\tilde{G}_{n-1}$.

We may assume that the function $a(w)$ is defined on almost the entire group $\tilde{G}_{n-1}$, taking $a(w)$ to be constant on each such coset. The equation (7.28) may then be written in the form

$$a(gg^0) = T_{g^0}^{-1} a(g) T_{g^0}, \tag{7.30}$$

where this equation holds for almost all $g \in \tilde{G}_{n-1}$ for every fixed g^0.

We put $g_1 = gg^0$; then (7.30) may be re-written as

$$a(g_1) = T^{-1}_{g^{-1}g_1} a(g) T_{g^{-1}g_1} = T_{g_1^{-1}} T_g a(g) T_{g^{-1}} T_{g_1},$$

whence

$$T_{g_1} a(g_1) T_{g_1}^{-1} = T_g a(g) T_g^{-1}. \tag{7.31}$$

The equation (7.31) holds for almost all pairs (g^0, g), g^0, $g \in \tilde{G}_{n-1}$, and consequently, by Fubini's theorem, for almost pairs $(gg^0, g) = (g_1, g)$, g_1, $g \in \tilde{G}_{n-1}$. Thus for almost all $g \in \tilde{G}_{n-1}$ the operator $T_g a(g) T_g^{-1}$ does not depend on g. We denote it by a. We have then

$$a(g) = T_g^{-1} a T_g \tag{7.32}$$

for almost all $g \in \tilde{G}_{n-1}$.

The function $a(g)$ is by definition constant on the right cosets of $\tilde{G}_{n-1}$ with respect to the stationary subgroup of the manifold $\mathfrak{B}$. We shall start from the point $(0, 0, \cdots, 1)$. Its stationary subgroup will be the set $\mathfrak{A}'$ of matrices $l = \| l_{pq} \| \in \tilde{G}_{n-1}$ such that the matrix $l^{*-1} = \| \hat{l}_{pq} \|$ satisfies the conditions

$$\hat{l}_{n-1,q} = 0 \quad \text{for} \quad q = 1, 2, \cdots, n-2, \quad \hat{l}_{n-1,n-1} > 0;$$

in consequence of this l satisfies the conditions:

$$l_{q,n-1} = 0, \quad \text{for} \quad q = 1, 2, \cdots, n-2, \quad l_{n-1,n-1} > 0.$$

Thus $\mathfrak{A}'$ is a subgroup of $\tilde{\mathfrak{A}}_{n-1}$.

On the right of (7.32) there stands a function of g which is continuous in the sense of convergence in the norm of $\mathfrak{H}$; it coincides for almost all g with a function $a(g)$ which is constant on cosets of the group $\tilde{G}_{n-1}$ with respect to the subgroup $\tilde{\mathfrak{A}}_{n-1}$. Hence, by Fubini's theorem, it follows that it is constant on almost all cosets with the possible exception of its values on a set of measure zero on the coset. By continuity it follows that $a(g)$ is constant on each coset without exception.

Taking in particular $g = l$ on the right of (7.32) we get:

$$T_l^{-1}\, a\, T_l = a, \tag{7.33}$$

and so the operator a commutes with all the operators T_l, $l \in \widetilde{\mathfrak{A}}_{n-1}$.

As already indicated, for $r = 1$ the operators T_{g^0} are operators of a unitary representation of the group $\widetilde{G}_{n-1}$ from the series d_{m0}; for $r > 1$ the formulae for the operators T_{g^0} differ from the formulae for the operators T_{g^0} of the unitary representation of the group $\widetilde{G}_{n-1}$ from the series $d_{m,\, r-1}$ solely by the factor

$$(\Lambda_{m+\tau-1} / |\Lambda_{m+\tau-1}|)^{\varepsilon_{m+\tau-1}}.$$

But for $r > 1$

$$\Lambda'_{m+\tau-1} = \tilde{g}'_{n-1} = x_{n-1,\,1} g^0_{1,\,n-1} + \cdots + x_{n-1,\,n-2} g^0_{n-2,\,n-1} + g^0_{n-1,\,n-1}. \tag{7.34}$$

If $g^0 = l \in \widetilde{\mathfrak{A}}_{n-1}$, then $g^0_{1,\,n-1} = \cdots = g^0_{n-2,\,n-1} = 0$, and Λ'_{m+r-1} is a constant number. Thus the operators T_l coincide with the operators T_l' of the unitary representation except possibly as regards a constant factor. Hence the operator a commutes with all the operators T_l', $l \in \widetilde{\mathfrak{A}}_{n-1}$. By our inductive hypothesis of the irreducibility of the representations of the $(n-1)$-th order group $\widetilde{\mathfrak{A}}_{n-1}$, the operator a is an operator of multiplication by a scalar, that is to say $a = \alpha \cdot 1$. But then it follows (7.32) that $a(g) = \alpha \cdot 1$ for almost all $g \in \widetilde{G}_{n-1}$. By (7.10) the operator A also coincide with $\alpha \cdot 1$.

And so, every bounded operator A on H which commutes with all the operators T_g, $g \in \mathfrak{A}_n$ is a multiple of the identity, and consequently T_g is an irreducible representation of the group $\mathfrak{A}_n$.

Case II. n an even number, $r > 0$. The space $\mathfrak{H}$ in which the representation T_g of the group $\widetilde{G}_n$ is realised, and the operators of the representation, are defined here precisely as for case I.

We have to prove in this case the irreducibility of the representation T_g of the group $\widetilde{\mathfrak{A}}_n$.

The proof is almost word for word the same as the proof for case I. We explain here the one point of the proof which requires somewhat different arguments.

As for case I, we consider in the group $\widetilde{\mathfrak{A}}_n$ the subgroup $\mathfrak{A}^0$ of matrices which satisfy the condition (7.12):

$$g_{nq} = 0 \quad \text{for} \quad q = 1, 2, \cdots, n-1.$$

In view of the fact that $n-1$ is an odd number, the real multiplier c in the formula (7.15):

$$g^0_{pq} = g_{pq} \cdot c, \quad p,\, q < n,$$

can always be chosen so that the determinant of the corresponding $(n-1)$-th order matrix g^0 is equal to 1. The multiplier c is of course determined uniquely by this condition. The sign of the multiplier c coincides with the sign of the determinant of the matrix g.

We put further $b = g_{nn} \cdot c$, as in case I. Here, however, the number b can be either positive or negative. The relation (7.29) which we reach means therefore that for almost all $w_1, \cdots, w_{n-1}$ the function $a(w_1, \cdots, w_{n-1})$ depends only on the ratios $w_1 : w_2 : \cdots : w_{n-1}$. Consequently one may regard this function as a function of points of the real projective space $\mathfrak{W}$, in which a point is defined as the ratio $w = (w_1 : w_2 : \cdots : w_{n-1})$.

The manifold $\mathfrak{W}$ may be regarded as a manifold of right cosets of the group G_{n-1} with respect to a certain subgroup of it. If we start from the point $(0 : 0 : \cdots : 1)$, then its stationary group will be the group $\mathfrak{A}_{n-1}$. Hence the proof of the lemma for case II is completed just as for case I.

Case III. n an even number, $r = 0$. By what was said in §7, sub-section 1, a representation of the group $\widetilde{G}_n$ is determined by $m = \frac{n}{2}$ positive integers $n_1, n_2, \cdots, n_m$ and $m - 1$ real numbers $\rho_1, \rho_2, \cdots, \rho_{m-1}$. The representation is realised in the space $\mathfrak{H}$ of functions

$$f(z) = f(\dot{z}x) = f(z_1, \cdots, z_m, x),$$

defined on the manifold $Z_{m,0}$ and analytic in each of the variables $z_1, \cdots, z_m$. The norm in the space $\mathfrak{H}$ is given by the formula

$$\|f\|^2 = \int |f(z_1, \ldots, z_m, x)|^2 \prod_{p=1}^{m} \frac{|\operatorname{Im} z_p|^{n_p-2}}{\Gamma(n_p-1)} d\mu(x) \cdot \\ \cdot dx_1 dy_1 \ldots dx_m dy_m \quad (z_p = x_p + iy_p), \tag{7.35}$$

and the operators of the representation T_g by the formula

$$T_g f(z) = f(z\overline{g}) \prod_{p=1}^{m} \left[\frac{\beta_p z_p + \delta_p}{\sqrt{|\alpha_p \delta_p - \beta_p \gamma_p|}} \right]^{-n_p} \cdot \\ \cdot \prod_{p=1}^{m-1} |\Lambda_p|^{i\rho_p} \cdot |\Lambda_1|^{n-2} \cdot |\Lambda_2|^{n-4} \cdots |\Lambda_{m-1}|^2. \tag{7.36}$$

We denote by z' the $(n-1)$-th order matrix whose elements are the elements z_{pq} of a matrix $z \in Z_{m,r}$ for $p < n$, $q < n$. Then $z' \in Z_{m-1,1}$, and the function $f(z)$ can be represented in the form

$$f(z) = f(z', x_{n1}, \cdots, x_{n,n-2}, z_m). \tag{7.37}$$

This function is of summable square with respect to the variables $x_{n1}, \cdots, x_{n,n-2}$. On the other hand, if we regard it as a function of z_m only, fixing the values of the remaining variables, then it will be analytic in z_m in the upper and lower half-planes separately. Hence its definition is equivalent to the definition of a pair of functions of which one is analytic in the upper and one in the lower half-plane.

In addition, for this function there must converge the integral

$$\int |f|^2 \frac{|\operatorname{Im} z_m|^{n_m-2}}{\Gamma(n_m-1)}\, dx_m\, dy_m.$$

We shall effect a Fourier transformation of the function $f(z', x_{n1}, \cdots, x_{n,n-2}, z_m)$ with respect to the variables $x_{n1}, \cdots, x_{n,n-2}, z_m$. First of all we make the following remarks.

If a function $f_1(z)$ of the complex variable $z = x + iy$ is analytic in the upper half-plane $y > 0$ and for some non-negative integral k

$$\int |f_1(z)|^2 \frac{y^{k-1}}{\Gamma(k)}\, dx\, dy < +\infty, \tag{7.38}$$

then the integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(z)\, e^{-itz}\, dx \tag{7.39}$$

converges and does not depend on y. If we denote this integral by $\phi_1(t)$, then the function $\phi_1(t)$ vanishes identically for $t < 0$. The function $f_1(z)$ is in turn expressible in terms of the function $\phi_1(t)$ by the formula

$$f_1(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \varphi_1(t)\, e^{itz}\, dx, \tag{7.40}$$

where

$$\int |f_1(z)|^2 \frac{y^{k-1}}{\Gamma(k)}\, dx\, dy = \frac{1}{2^k} \int_0^{\infty} |\varphi_1(t)|^2 t^{-k}\, dt. \tag{7.41}$$

In view of this there holds good an isometric correspondence between functions $f_1(z)$, analytic in the upper half-plane with the norm

$$\| f_1 \|^2 = \int_{-\infty}^{+\infty} \int_0^{\infty} |f_1(z)|^2 \frac{y^{k-1}}{\Gamma(k)}\, dy\, dx$$

and functions $\phi_1(t)$, defined on the half-line $0 \leq t < +\infty$, with the norm

$$\| \varphi_1 \|^2 = \frac{1}{2^k} \int_0^{\infty} |\varphi_1(t)|^2 t^{-k}\, dt.$$

We indicate the proof of this assertion. If $k > 0$, then the convergence of the integral (7.38) implies the convergence (for almost all y) of the integral

$$\int_{-\infty}^{+\infty} |f_1(z)|^2\, dx = \int_{-\infty}^{+\infty} |f_1(x, y)|^2\, dx.$$

It is therefore possible to define a function

$$\varphi_1(t, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(x, y)\, e^{-ixt}\, dx. \tag{a}$$

From the Plancherel formula we have

$$\int_{-\infty}^{+\infty} |\varphi_1(t, y)|^2 \, dt = \int_{-\infty}^{+\infty} |f_1(x, y)|^2 \, dx,$$

and hence

$$\int_0^{\infty} dy \int_{-\infty}^{+\infty} |\varphi_1(t, y)|^2 \frac{y^{k-1}}{\Gamma(k)} \, dt = \int_0^{\infty} dy \int_{-\infty}^{+\infty} |f_1(x, y)|^2 \frac{y^{k-1}}{\Gamma(k)} \, dx < +\infty. \qquad \text{(b)}$$

On the other hand, the function $f_1(z) = f_1(x, y)$, being analytic, satisfies the differential equation

$$\frac{\partial f_1}{\partial x} + i \frac{\partial f_1}{\partial y} = 0.$$

Using (a), we derive from this the differential equation for the function $\phi_1(t, y)$:

$$it\phi_1 + i \frac{\partial \phi_1}{\partial y} = 0.$$

Hence

$$\phi_1(t, y) = e^{-ty} \phi_1(t), \qquad \text{(c)}$$

where $\phi_1(t)$ is a certain function depending only on t.

Substituting in (b) we get

$$\int_0^{\infty} dy \int_{-\infty}^{+\infty} |\varphi_1(t)|^2 \frac{y^{k-1}}{\Gamma(k)} e^{-2ty} \, dt < +\infty.$$

It follows immediately from the convergence of the integral just written down that $\phi_1(t) \equiv 0$ for $t < 0$.

Multiplying both sides of the equation (a) by e^{ty} and using the expression (c) we get

$$\varphi_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(z) e^{-itz} \, dx \quad (z = x + iy).$$

The function $f_1(z)$ can in turn be expressed in terms of the function $\phi_1(t)$ by the formula

$$f_1(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \varphi_1(t) e^{itz} dt.$$

Furthermore, by (b) and (c),

$$\int_0^{\infty} dy \int_{-\infty}^{+\infty} |f_1(x, y)|^2 \frac{y^{k-1}}{\Gamma(k)} \, dx = \int_0^{\infty} |\varphi_1(t)|^2 \, dt \int_0^{\infty} e^{-2ty} \frac{y^{k-1}}{\Gamma(k)} \, dy.$$

Making the change of variable $2ty = u$ in the second integral we have

$$\int_0^{\infty} e^{-2ty} \frac{y^{k-1}}{\Gamma(k)} \, dy = \frac{\Gamma^{-1}(k)}{(2t)^k} \int_0^{\infty} e^{-u} u^{k-1} du = \frac{1}{(2t)^k}.$$

Thus

$$\int_{y>0} |f_1(z)|^2 \frac{y^{k-1}}{\Gamma(k)} \, dx \, dy = \frac{1}{2^k} \int_0^{\infty} |\varphi_1(t)|^2 t^{-k} dt.$$

Conversely, if a function $\phi_1(t)$, defined on a half-line $0 \leq t < \infty$, is such that for it there converges the integral

$$\int_0^\infty |\varphi_1(t)|^2 t^{-k} dt,$$

then the function $f_1(z) = f_1(x, y)$, given by the formula (7.40), is analytic for all z in the half-plane $\operatorname{Im} z > 0$.

If finally $k = 0$, then $y^{k-1}/\Gamma(k)$ is to be regarded as a delta-function. The integral (7.38) then takes the form:

$$\int_{-\infty}^{+\infty} |f_1(x)|^2 dx,$$

where $f_1(x)$ is the boundary value of the function $f_1(z)$ on the real axis. Hence we may define a function

$$\varphi_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(x) e^{-itx} dx.$$

As is known (see, for example, [4]), $\phi_1(t) \equiv 0$ for $t < 0$ and

$$\int_{-\infty}^{+\infty} |f_1(x)|^2 dx = \int_0^\infty |\varphi_1(t)|^2 dt.$$

Furthermore,

$$f_1(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \varphi_1(t) e^{itz} dt.$$

From this one can also get an expression for the function $\phi_1(t)$ in the form of the integral (7.39).

An assertion analogous to what has just been proved is valid for a function $f_2(z)$ analytic in the lower half-plane $y < 0$. The function $\phi_2(t)$ which corresponds to it will vanish identically for $t > 0$.

And so, we are finally in a position to associate with every pair of such functions $f_1(z)$ and $f_2(z)$ a function $\phi(t) = \phi_1(t) + \phi_2(t)$ which is defined on the real axis. This correspondence is given by the formula

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_1(z) e^{-itz} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f_2(z) e^{-itz} dx,$$

where in the first integral the integration is taken along the line $y = c > 0$, and in the second along the line $y = -c < 0$.

We shall denote by $f(z)$ the pair of functions $f_1(z)$ and $f_2(z)$. The last formula may then be written in the form

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int f(z) e^{-itz} dx, \tag{7.42}$$

where the integration is along the line $y = c$ and $y = -c$.

The transformation (7.42) constitutes an isometric mapping of the space of functions $f(z)$ of a complex variable, analytic in the upper and in the lower half-plane, with the norm

$$\| f \|^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} | f(z) |^2 \frac{| y |^{k-1}}{\Gamma(k)} \, dy \, dx,$$

on to the space of functions $\phi(t)$, defined on the real axis, with the norm

$$\| \varphi \|^2 = \frac{1}{2^k} \int_{-\infty}^{+\infty} | \varphi(t) |^2 | t |^{-k} dt.$$

We now return to the functions $f(\dot{z}x)$ of the space $\mathfrak{H}$. By what was said above, these functions admit the Fourier transformation:

$$\varphi(z', w_1, \ldots, w_{n-2}, \xi) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int f(z', x_{n_1}, \ldots, x_{n, n-2}, z_m) \cdot$$
$$\cdot e^{-i(x_{n1} w_1 + \cdots + x_{n, n-2} w_{n-2} + z_m \xi)} dx_{n1} \ldots dx_{n, n-2} \, dx_m. \tag{7.43}$$

This transformation constitutes an isometric mapping of the space $\mathfrak{H}$ on to the space H of function $\phi(z', w_1, \cdots, w_{n-2}, \xi)$, defined for $z' \in Z_{m-1, 1}$, $-\infty < w_k < +\infty$, $-\infty < \xi < +\infty$, with the norm

$$\| \varphi \|^2 = \frac{1}{2^{n_m - 1}} \int | \varphi(z', w_1, \ldots, w_{n-2}, \xi) |^2 \prod_{p=1}^{m-1} \frac{| \operatorname{Im} z_p |^{n_p - 2}}{\Gamma(n_p - 1)} | \xi |^{1 - n_m} \cdot$$
$$\cdot dz' dw_1 \ldots dw_{n-2} \, d\xi. \tag{7.44}$$

We introduce in place of ξ a new variable w_{n-1}, connected with ξ by the relation

$$\xi = w_{n-1} + x_{n-1,1} w_1 + \cdots + x_{n-1, n-2} w_{n-2}. \tag{7.45}$$

The space H may then be regarded as a space of functions $\phi(z', w_1, \cdots, w_{n-1})$ with the norm

$$\| \varphi \|^2 = \frac{(n_m - 1)!}{2^{n_m - 1}} \int | \varphi(z', w_1, \ldots, w_{n-1}) |^2 \prod_{p=1}^{m-1} \frac{| \operatorname{Im} z_p |^{n_p - 2}}{\Gamma(n_p - 1)} \cdot$$
$$\cdot | w_{n-1} + x_{n-1, 1} w_1 + \ldots + x_{n-1, n-2} w_{n-2} |^{1 - n_m} dz' dw_1 \ldots dw_{n-1}. \tag{7.46}$$

The formula (7.43) now assumes the form

$$\varphi(z', w_1, \ldots . w_{n-1}) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int f(z', x_{n1}, \ldots, x_{n, n-2}, z_m) \cdot$$
$$\cdot e^{-i\left(\sum_{s=1}^{n-2} x_{ns} w_s + z_m \left(w_{n-1} + \sum_{s=1}^{n-2} x_{n-1, s} w_s\right)\right)} dx_{n1} \ldots dx_{n, n-2} \, dx_m$$

or

$$\varphi(z', w_1, \dots, w_{n-1}) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int f(z', x_{n1}, \dots, x_{n,n-2}, z_m) \cdot$$

$$\cdot e^{-i\left(\sum\limits_{s=1}^{n-2} (x_{ns}+z_m x_{n-1,s}) w_s + z_m w_{n-1}\right)} dx_{n1} \dots dx_{n,n-2}\, dx_m. \qquad (7.47)$$

We shall put this formula into a slightly different form. For this purpose we observe that the elements of the bottom row of a matrix $z \in Z_{m,0}$ may be expressed in terms of the elements of the matrices $\dot{z} \in \dot{Z}_{m,0}$ and $x \in X_{m,0}$, which appear in the representation $z = \dot{z}x$, according to the formulae

$$z_{nq} = x_{nq} + z_m x_{n-1,q}, \quad q = 1, \cdots, n-2, \quad z_{n,n-1} = z_m. \qquad (7.48)$$

We put $u_q = \operatorname{Re} z_{nq}$, $q = 1, \cdots, n-1$, i.e.,

$$u_q = x_{nq} + x_m x_{n-1,q}, \quad q = 1, \cdots, n-2, \quad u_{n-1} = x_m. \qquad (7.49)$$

A function $f(z) \in \mathfrak{H}$ may be put in the form

$$f(z) = f(z', u_1, \cdots, u_{n-2}, z_m).$$

Changing from the variables $x_{n1}, \cdots, x_{n,n-2}, x_m$ to the new variables $u_1, u_2, \cdots, u_{n-1}$ in (7.47) we get:

$$\varphi(z', w_1, \dots, w_{n-1}) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int f(z', u_1, \dots, u_{n-2}, z_m) \cdot$$

$$\cdot\, e^{-i(z_{n1}w_1 + \dots + z_{n,n-2}w_{n-2} + z_{n,n-1}w_{n-1})}\, du_1 \dots du_{n-1}. \qquad (7.50)$$

We consider the transformation

$$\psi(z', w_1, \dots, w_{n-1}) = \left| w_{n-1} + \sum_{s=1}^{n-2} x_{n-1,s} w_s \right|^{-\frac{n_m-1}{2}} \varphi(z', w_1, \dots, w_{n-1}). \qquad (7.51)$$

It constitutes an isometric mapping of the space H on to a space H_1 of functions $\psi(z', w_1, \cdots, w_{n-1})$ with the norm

$$\|\psi\|^2 = \frac{1}{2^{n_m-1}} \int |\psi(z', w_1, \dots, w_{n-1})|^2 \prod_{p=1}^{m-1} \frac{|\operatorname{Im} z_p|^{n_p-2}}{\Gamma(n_p-1)}\, dz'\, dw_1 \dots dw_{n-1}. \qquad (7.52)$$

We remark that if $\phi \to \psi$ under the mapping (7.51), then the action of the operator T_g on the function ψ in the space H_1 is expressible by the formula

$$T_g\psi = \left| w_{n-1} + \sum_{s=1}^{n-2} x_{n-1,s} w_s \right|^{-\frac{n_m-1}{2}} T_g\varphi. \qquad (7.53)$$

Let A be a bounded operator in the space H_1 which commutes with all the operators T_g, $g \in \mathfrak{A}_n$. We have to show that this operator is a multiple of the identity operator.

We consider in the group G_n the matrices $x^0 \in Z_{m,0}$ which satisfy the condition (7.6): $x^0_{pq} = 0$ for $p < n$.

In the space $\mathfrak{H}$ the operator T_{x^0} is the displacement operator:

$$T_{x^0}f(z) = f(zx^0). \tag{7.54}$$

On the other hand it follows from the condition (7.6) that under the displacement $z \to zx^0$ the elements of the matrix z' remain unchanged, and $z_{nq} \to z_{nq} + x^0_{nq}$. Hence, by the definition of the quantities u_q, $u_q \to u_q + x^0_{nq}$ and $x_m \to x_m + x_{n,n-1}$. Hence (7.54) can be re-written in the form

$$\begin{aligned} &T_{x^0}f(z', u_1, \ldots, u_{n-2}, z_m) = \\ &= f(z', u_1 + x^0_{n1}, \ldots, u_{n-2} + x^0_{n,\,n-2}, z_m + x^0_{n,\,n-1}). \end{aligned} \tag{7.55}$$

Hence, by (7.50), the operator T_{x^0} in the space H will have the form:

$$T_{x^0}\varphi(z', w_1, \ldots, w_{n-1}) = e^{i\left(x^0_{n1}w_1 + \cdots + x^0_{n,\,n-1}w_{n-1}\right)}\varphi(z', w_1, \ldots, w_{n-1}). \tag{7.56}$$

In view of (7.53) this operator will have the same form also in the space H_1. From this we deduce, as for case I, that in the space H_1 the operator A has the form:

$$A\psi(z', w_1, \ldots, w_{n-1}) = a(w_1, \ldots, w_{n-1})\psi(z', w_1, \ldots, w_{n-1}),$$

where $a(w_1, \cdots w_{n-1})$ is an operator in the space of functions $f(z')$ with norm

$$\|f\|^2 = \int |f(z')|^2 \prod_{p=1}^{m-1} \frac{|\operatorname{Im} z_p|^{n_p-2}}{\Gamma(n_p-1)}\, dz',$$

defined for almost all $w_1, \cdots, w_{n-1}$ and uniformly bounded with respect to these variables.

We consider in the group $\tilde{\mathfrak{A}}_n$ the subgroup $\mathfrak{A}^0$ of matrices which satisfy the supplementary condition (7.12):

$$g_{nq} = 0 \quad \text{for} \quad q = 1, 2, \cdots, n-1,$$

and ascertain the form of operators T_g, $g \in \mathfrak{A}^0$.

We put, as for case II,

$$g^0_{pq} = g_{pq} \cdot c, \quad p, q < n,$$

where the factor c is chosen so that the $(n-1)$-th order matrix g^0 is unimodular. We put further $b = g_{nn} \cdot c$. It follows from (7.13) and (7.14) that under the transformation $z \to z\bar{g}$ the matrix z' goes into $z'\overline{g^0}$, and z_{nq} into $\hat{z}_{nq}$, where

$$b\hat{z}_{nq} = \sum_{s=1}^{n-1} g^0_{sq} z_{ns}. \tag{7.57}$$

Putting $\hat{u}_q = \operatorname{Re}\hat{z}_{nq}$, we have also

$$b\hat{u}_q = \sum_{s=1}^{n-1} g^0_{sq} u_s. \tag{7.58}$$

Let us calculate $a(z, g)$. The expressions Λ_p which enter into the formula for $a(z, g)$ are calculated by means of the formulae

$$\Lambda_p = \frac{\tilde{g}_{2(p-1)+1}}{\tilde{g}_{2p+1}}, \quad p = 1, \ldots, m-1, \tag{7.59}$$

where $\tilde{g}_m$ denotes a principal minor of the matrix $\tilde{g} = xg$.

We represent a matrix $z' \in Z_{m-1,1}$ in the form

$$z' = \dot{z}' x', \quad \dot{z}' \in \dot{Z}_{m-1,1}, \quad x' \in X_{m-1,1}.$$

We put

$$\Lambda'_p = \frac{\tilde{g}'_{2(p-1)+1}}{\tilde{g}'_{2p+1}},$$

where the $\tilde{g}'_m$ denote principal minors of the matrix $\tilde{g}' = x' g^0$. As for case I, it is not difficult to verify that these expressions differ from the corresponding expressions Λ_p by a constant factor, which is a power of the number c.

Let us furthermore calculate the factor $\dfrac{\beta_m z_m + \delta_m}{\sqrt{|\alpha_m \delta_m - \beta_m \gamma_m|}}$ which enters into $a(z, g)$. For this we represent the matrix xg in the form $xg = \hat{k} x_1$, where $\hat{k} = \| \hat{k}_{pq} \| \in \hat{K}_m$ and $x_1 \in X_m$. Then, in accordance with the formulae of §6,

$$\alpha_m = k_{n-1,n-1}, \quad \beta_m = k_{n-1,n}, \quad \gamma_m = k_{n,n-1}, \quad \delta_m = k_{nn}.$$

But a direct calculation gives us

$$k_{n-1,n-1} = \sum_{s=1}^{n-2} x_{n-1,s} g_{s,n-1} + g_{n-1,n-1}, \quad k_{n-1,n} = 0,$$
$$k_{n,n-1} = \sum_{s=1}^{n-2} x_{ns} g_{s,n-1} + g_{n,n-1}, \qquad k_{nn} = g_{nn}.$$

Thus

$$\frac{\beta_m z_m + \delta_m}{\sqrt{|\alpha_m \delta_m - \beta_m \gamma_m|}} = \sqrt{g_{nn}} \left| \sum_{s=1}^{n-2} x_{n-1,s} g_{s,n-1} + g_{n-1,n-1} \right|^{-\frac{1}{2}} =$$
$$= \sqrt{g_{nn} |c|} \cdot \left| \sum_{s=1}^{n-2} x_{n-1,s} g^0_{s,n-1} + g^0_{n-1,n-1} \right|^{-\frac{1}{2}}. \tag{7.60}$$

Furthermore,

$$|\Lambda_m| = |\alpha_m \delta_m - \beta_m \gamma_m| = g_{nn} |c|^{-1} \cdot \left| \sum_{s=1}^{n-2} x_{n-1,s} g^0_{s,n-1} + g^0_{n-1,n-1} \right|.$$

The formula for $a(z, g)$ hence takes the following form:

$$a(z, g) = a_0(g_{nn}) \, a'(z', g^0), \tag{7.61}$$

where $a_0(g_{nn})$ is a function of the form (7.20), and $a'(z', g^0)$ is a function of z'

and g^0 of the form:

$$\alpha'(z', g^0) = \prod_{p=1}^{m-1} \left[\frac{\beta_p z_p + \delta_p}{\sqrt{|\alpha_p \delta_p - \beta_p \gamma_p|}}\right]^{-n_p} \cdot \prod_{p=1}^{m-1} |\Lambda'_p|^{i\rho_p} \cdot$$

$$\cdot |\Lambda'_1|^{n-2} |\Lambda'_2|^{n-4} \dots |\Lambda'_{m-1}|^2 \cdot \left|\sum_{s=1}^{n-2} x_{n-1,s}\, g^0_{s,n-1} + g^0_{n-1,n-1}\right|^{\frac{n_m}{2}}. \quad (7.62)$$

The formula for the operator T_g, $g \in \mathfrak{U}^0$, in the space $\mathfrak{H}$ has thus the form:

$$T_g f(z', u_1, \dots, u_{n-2}, z_m) = \alpha_0(g_{nn})\, \alpha'(z', g^0) f(z'\overline{g^0}, \hat{u}_1, \dots, \hat{u}_{n-2}, \hat{z}_m). \quad (7.63)$$

Let us see how the operator T_g acts on the function $\phi(z', w_1, \cdots, w_{n-1})$ from H. We have

$$T_g \varphi(z', w_1, \dots, w_{n-1}) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \alpha_0(g_{nn})\, \alpha'(z', g^0) \int f(z'\overline{g^0}, \hat{u}_1, \dots, \hat{u}_{n-2}, \hat{z}_m) \cdot$$

$$\cdot e^{-i(z_{n1}w_1 + \dots + z_{n,n-1}w_{n-1})}\, du_1 \dots du_{n-1}.$$

But, by (7.57),

$$z_{n1}w_1 + \dots + z_{n,n-1}w_{n-1} = \hat{z}_{n1}\hat{w}_1 + \dots + \hat{z}_{n,n-1}\hat{w}_{n-1},$$

where the quantities $\hat{w}_q$ are given by

$$\frac{1}{b}\hat{w}_q = \sum_{s=1}^{n-1} \hat{g}_{sq} w_s, \quad (7.64)$$

where the matrix $\hat{g} = \| \hat{g}_{sq} \|$ in the inverse of the matrix which is adjoint to g^0, that is to say $\hat{g} = (g^0)^{*-1}$.

Hence, going over to the new integration-variables $\hat{u}_1, \cdots, \hat{u}_{n-1}$ and denoting by Δ the Jacobian of the transformation (7.58), we obtain:

$$T_g \varphi(z', w_1, \dots, w_{n-1}) = \frac{\Delta}{(2\pi)^{\frac{n-1}{2}}} \alpha_0(g_{nn})\, \alpha'(z', g^0) \int f(z'\overline{g^0}, \hat{u}_1, \dots, \hat{u}_{n-2}, \hat{z}_m) \cdot$$

$$\cdot e^{-i(\hat{z}_{n1}\hat{w}_1 + \dots + \hat{z}_{n,n-1}\hat{w}_{n-1})}\, d\hat{u}_1 \dots d\hat{u}_{n-1}$$

or

$$T_g \varphi(z', w_1, \dots, w_{n-1}) = \frac{\Delta}{(2\pi)^{\frac{n-1}{2}}} \alpha_0(g_{nn})\, \alpha'(z', g^0) \cdot \varphi(z'\overline{g^0}, \hat{w}_1, \dots, \hat{w}_{n-1}). \quad (7.65)$$

Finally we write down the formula for the operator T_g in the space H_1. Let, under the transformation $z' \to z'\overline{g^0}$, the elements $x_{n-1,q}$ of the bottom row of the matrix z' be transformed into elements $\hat{x}_{n-1,q}$. Then, in virtue of (7.65), (7.51) and (7.53), we obtain the following formula for the operator T_g in the space H_1:

$$T_g\psi(z', w_1, \ldots, w_{n-1}) = \frac{\Delta}{(2\pi)^{\frac{n-1}{2}}}\, \alpha_0(g_{nn})\, \alpha'(z', g^0)\,\cdot$$

$$\cdot \left|\frac{\sum\limits_{s=1}^{n-2} \hat{x}_{n-1,s}\hat{w}_s + \hat{w}_{n-1}}{\sum\limits_{s=1}^{n-2} x_{n-1,s}w_s + w_{n-1}}\right|^{\frac{n_m-1}{2}} \cdot\ \psi(z'\bar{g}^0, \hat{w}_1, \ldots, \hat{w}_{n-1}). \tag{7.66}$$

We transform the expression

$$\frac{\sum\limits_{s=1}^{n-2} \hat{x}_{n-1,s}\hat{w}_s + \hat{w}_{n-1}}{\sum\limits_{s=1}^{n-2} x_{n-1,s}w_s + w_{n-1}}.$$

As is easy to see, the quantities $\hat{x}_{n-1,s}$ can be expressed in terms of the elements of the bottom row of the matrix $g' = z'g^0$ according to the formulae

$$\hat{x}_{n-1,s} = \frac{g'_{n-1,s}}{g'_{n-1,n-1}}. \tag{7.67}$$

Here

$$g'_{n-1,n-1} = \sum_{s=1}^{n-2} x_{n-1,s}\, g^0_{s,n-1} + g^0_{n-1,n-1}. \tag{7.68}$$

Hence we have, by (7.64) and (7.67):

$$\sum_{s=1}^{n-2} \hat{x}_{n-1,s}\hat{w}_s + \hat{w}_{n-1} = \frac{1}{g'_{n-1,n-1}} \sum_{s=1}^{n-1} g'_{n-1,s}\,\hat{w}_s =$$

$$= \frac{b}{g'_{n-1,n-1}} \sum_{s=1}^{n-1}\left(\sum_{p=1}^{n-1} x_{n-1,p}\, g^0_{ps}\right)\left(\sum_{q=1}^{n-1} \hat{g}_{sq}\, w_s\right). \tag{7.69}$$

But $g = (\hat{g}^0)^{*-1}$, and hence the elements of the matrices g^0 and $\hat{g}$ are connected by the relation

$$\sum_{s=1}^{n-1} g^0_{ps}\hat{g}_{qs} = \delta_{pq},$$

where δ_{pq} is the Kronecker symbol. Hence (7.69) takes the form:

$$\sum_{s=1}^{n-2} \hat{x}_{n-1,s}\hat{w}_s + \hat{w}_{n-1} = \frac{b\left(\sum\limits_{s=1}^{n-2} x_{n-1,s}w_s + w_{n-1}\right)}{\sum\limits_{s=1}^{n-2} x_{n-1,s}\, g^0_{s,n-1} + g^0_{n-1,n-1}},$$

and so

$$\frac{\sum\limits_{s=1}^{n-2}\hat{x}_{n-1,\,s}\hat{w}_s+\hat{w}_{n-1}}{\sum\limits_{s=1}^{n-2}x_{n-1,\,s}w_s+w_{n-1}}=\frac{b}{\sum\limits_{s=1}^{n-2}x_{n-1,\,s}\,g^0_{s,\,n-1}+g^0_{n-1,\,n-1}}. \tag{7.70}$$

From (7.62) and (7.70) we obtain:

$$\cdot\left|\frac{\sum\limits_{s=1}^{n-2}\hat{x}_{n-1,\,s}\hat{w}_s+\hat{w}_{n-1}}{\sum\limits_{s=1}^{n-2}x_{n-1,\,s}w_s+w_{n-1}}\right|^{\frac{n_m-1}{2}}\cdot\alpha'(z',\,g^0)=\prod_{p=1}^{m-1}\left[\frac{\beta_p z_p+\delta_p}{\sqrt{|\alpha_p\delta_p-\beta_p\gamma_p|}}\right]^{-n_p}\cdot$$

$$\cdot\prod_{p=1}^{m-1}|\Lambda'_p|^{i\rho_p}|\Lambda'_1|^{n-2}|\Lambda'_2|^{n-4}\dots|\Lambda'_{m-1}|^2\cdot\left|\sum_{s=1}^{n-2}x_{n-1,\,s}g^0_{s,\,n-1}+g^0_{n-1,\,n-1}\right|^{\frac{1}{2}}\cdot b^{\frac{n_m-1}{2}}. \tag{7.71}$$

But, as was proved above,

$$\left|\sum_{s=1}^{n-2}x_{n-1,\,s}\,g^0_{s,\,n-1}+g^0_{n-1,\,n-1}\right|=\frac{|c|}{g_{nn}}|\Lambda_m|.$$

On the other hand $|\Lambda_1\cdot\Lambda_2\cdots\Lambda_m|=1$, and hence

$$\left|\sum_{s=1}^{n-2}x_{n-1,\,s}\,g^0_{s,\,n-1}+g^0_{n-1,\,n-1}\right|^{\frac{1}{2}}=k|\Lambda'_1|^{-\frac{1}{2}}|\Lambda'_2|^{-\frac{1}{2}}\dots|\Lambda'_{m-1}|^{-\frac{1}{2}}, \tag{7.72}$$

where k is a constant factor which is a power of the number c.

Thus the formula (7.66) for the operator T_g in the space H_1 assumes the form

$$T_g\psi(z',\,w_1,\,\dots,\,w_{n-1})=$$
$$=\frac{1}{(2\pi)^{\frac{n-1}{2}}}\alpha_1(g_{nn})\tilde{\alpha}(z',\,g^0)\,\psi(z'\bar{g}^0,\,\hat{w}_1,\,\dots,\,\hat{w}_{n-1}). \tag{7.73}$$

Here $a_1(g_{nn})$ is again some function of the form (7.20), and

$$\tilde{\alpha}(z',\,g^0)=\prod_{p=1}^{m-1}\left[\frac{\beta_p z_p+\delta_p}{\sqrt{|\alpha_p\delta_p-\beta_p\gamma_p|}}\right]^{-n_p}\cdot\prod_{p=1}^{m-1}|\Lambda'_p|^{i\rho_p}\,\cdot$$
$$\cdot|\Lambda'_1|^{(n-1)-\frac{3}{2}}|\Lambda'_2|^{(n-1)-\frac{7}{2}}\dots|\Lambda'_{m-1}|^{\frac{3}{2}}. \tag{7.74}$$

The function $\tilde{a}(z',g^0)$ coincides, as is easy to see, with a function giving a unitary representation of the group G_{n-1} from the series $d_{m-1,1}$.

And so, for $g\in\mathfrak{A}^0$,

$$T_g\psi(z',\,w_1,\,\dots,\,w_{n-1})=\frac{\alpha_1(g_{nn})}{(2\pi)^{\frac{n-1}{2}}}T_{g^\circ}\psi(z',\,\hat{w}_1,\,\dots,\,\hat{w}_{n-1}), \tag{7.75}$$

where T_{g^0} is a unitary representation of the unimodular group G_{n-1} of the $(n-1)$-th order from the series $d_{m-1,1}$, this representation being given by the equation

$$T_{g^0} f(z') = \tilde{a}(z', g^0) f(z', g^0), \tag{7.76}$$

and $\hat{w}_1, \cdots, \hat{w}_{n-1}$ are expressed by the formula (7.64).

The proof of the fact that an operator A which commutes with all the operators T_g, $g \in \tilde{\mathfrak{U}}_n$, is necessarily a multiple of the identity operator, is now completed precisely as for case II.

To complete the proof of Lemma 1 it remains only to consider the case $n = 2$.

For $n = 2$ we have two series of representations of the group $\tilde{G}_2 : d_{0,2}$ and $d_{1,0}$. Here z' is absent. Hence, in the case of the series $d_{0,2}$ and likewise in the case of the series $d_{1,0}$ the formula for the operator A in the appropriate functional space assumes the form

$$Af(w) = a(w) f(w), \tag{7.77}$$

where $a(w)$ is a scalar function which is defined and bounded for almost all w. The relation (7.29) means in this case that $a(bw) = a(w)$. Thus $a(w)$ is a constant. Hence the irreducibility of the representation is proved for this case also.

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REPRESENTATIONS OF THE GROUP OF ROTATIONS OF 3-DIMENSIONAL SPACE AND THEIR APPLICATIONS

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The present article is devoted to a study of the representations of the group of rotations of 3-dimensional space. The reader is expected only to be familiar with the foundations of linear algebra, for example, with the contents of the first two chapters of the book of I. M. Gel'fand, *Lectures on linear algebra*.

The theory of representations, and in particular the theory of representations of the 3-dimensional rotations group, is extensively used in quantum mechanics. In the present article, all material is collected which is necessary for quantum mechanical applications. On the other hand, a study of the 3-dimensional rotation group can serve as a good introduction to the general theory of representations of Lie groups. In this example, we see very clearly displayed the connections between the theory of representations and other branches of mathematics (spherical functions, tensors, and so on); in the general case, these connections have not always been given sufficient attention.

For the most part, the contents of the present article are well known. Certain new results are contained in the final sections (§§7–9).

TABLE OF CONTENTS

§1. **The group of rotations of 3-dimensional space (the orthogonal group)** 208

1. Definition of the rotation group. 2. Introduction of parameters in the rotation group. 3. Invariant integration. 4. Connection between the rotation group and the group of unitary matrices of order 2. 5. Definition of representations of the rotation group.

§2. **Infinitesimal rotations and description of the irreducible representations of the rotation group** 220

1. Definition of the matrices A_k corresponding to infinitesimal rotations. 2. Relations among the matrices A_k. 3. The form of an irreducible representation. 4. Decomposition of a representation into irreducible parts. 5. Examples of representations. Appendix to §2. Proof of the differentiability of the matrix T_g.

§3. **Spherical functions and representations of the rotation group** 236

1. Definition of spherical functions. 2. Differential operators corresponding to infinitesimal rotations. 3. The differential equation of spherical functions. 4. An explicit expression for spherical functions. 5. Expansion of functions on the sphere in spherical functions.

§4. **The product of representations** 245

1. Definition of the product of representations. 2. Transformations in a product of representations which correspond to infinitesimal rotations. 3. The product of two irreducible representations. 4. Decomposition of

the product of two irreducible representations when one of them has weight *l* or *1/2*.

§5. Tensors and tensor representations 254

1. The principal algebraic operations on tensors. Invariant subspaces. 2. Computation of the weights of the irreducible representations into which a tensor representation is decomposed. 3. Decomposition of a tensor representation into multiples of irreducible representations. Tensors of rank *3*.

§6. Spinors and spinor representations 266

1. Definition of a spinor and of a spinor representation. 2. Symmetric spinors. Existence of irreducible representations for every half-integer weight *l*. 3. The principal operations on spinors. 4. Decomposition of a spinor representation into irreducible parts.

§7. Matrix elements of an irreducible representation (generalized spherical functions) 271

1. The operators U_g. 2. Differential operators corresponding to infinitesimal rotations. 3. Dependence of the matrix elements on Euler's angles ϕ_1 and ϕ_2. 4. Generalized spherical functions. 5. The addition theorem for matrix elements. 6. Expansion of functions on the rotation group in series of generalized spherical functions. Appendix to §7. Recurrence relations among generalized spherical functions.

§8. Expansion of vector and tensor fields 288

1. Expansion of vector functions. 2. Expansion of arbitrary quantities. 3. Example. A field of tensors of rank *2*. 4. Solution of Maxwell's equations.

§9. Equations which are invariant under rotations 303

1. Definition of invariant equations. 2. A restatement of the conditions for invariance. 3. Determination of the matrices L_1, L_2, L_3. 4. Solution of invariant equations. 5. Solution of Dirac's equations.

§1. The group of rotations of *3*-dimensional space

(the orthogonal group)

1. Definition of the rotation group. We consider all rotations of *3*-dimensional space around a fixed point (the origin of co-ordinates). Let g_1 and g_2 be two rotations. Then their product g is the rotation obtained by first carrying out the rotation g_2 and then the rotation g_1.* We write this in the following way: $g=g_1g_2$. It is not difficult to verify that the set G of all rotations forms a group, that is, that all of the axioms for a group are satisfied under this definition of the product of two rotations. The identity e of the group, or, as we shall say, the identity rotation, is naturally the rotation through the angle *0*.

If x is a vector issuing from the origin, then the rotation g carries it into a vector x'. We shall denote this as follows:

---

* This order is ordinarily used in multiplying linear transformations.

$$x' = gx. \tag{1}$$

We shall now explain how to give an analytic description of a rotation. Choose a fixed orthogonal system of co-ordinates in 3-dimensional space. In this co-ordinate system, the rotation is given by the formulas

$$x'_i = \sum_{k=1}^{3} g_{ik}x_k, \tag{2}$$

where the x_k are the co-ordinates of the vector x and the x'_i are the co-ordinates of the vector x'. The matrix $\| g_{ik} \|$ defines the given rotation. We shall denote this matrix by the same letter, g, that denotes the rotation itself. We now describe the conditions that the numbers g_{ik} must satisfy. Since a rotation does not alter either distances or angles, it does not alter scalar products. Thus, if $x' = gx$ and $y' = gy$, then

$$\sum_{i=1}^{3} x'_i y'_i = \sum_{k=1}^{3} x_k y_k. \tag{3}$$

Substituting in the left-hand side of (3) the values of x'_i and y'_i given by formula (2), we obtain

$$\sum_{i,k,l} g_{ik}g_{il}x_k y_l = \sum_{k=1}^{3} x_k y_k.$$

Equating the coefficients of the $x_k y_l$ on the left and right sides of the last equation, we obtain

$$\sum_{i=1}^{3} g_{ik}g_{il} = \delta_{kl}, \tag{4}$$

where the symbol δ_{kl} is taken to be equal to 1 if $k=l$ and 0 if $k \neq l$. The equality (4) can be written in matrix form, as we now show. On the right side of the equality, we have the elements of the identity matrix e, and on the left side we have the elements of the product $g'g$, where g' is the transpose of the matrix g. Thus

$$g'g = e \tag{5}$$

or equivalently

$$g' = g^{-1}. \tag{5'}$$

Matrices which satisfy the equality (5) are called *orthogonal matrices*. Taking the determinant of both sides of the equality (5), we obtain $\mathrm{Det}(g') \cdot \mathrm{Det}(g) = 1$, hence $[\mathrm{Det}(g)]^2 = 1$, and consequently

$$\mathrm{Det}(g) = \pm 1. \tag{6}$$

Every rotation is a rotation through an angle ϕ about some axis of rotation. If the axis of rotation is the z-axis, then the matrix of this rotation has the form

$$\begin{Vmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{7}$$

Since the determinant of the matrix (7) is equal to 1, the determinant of the rotation, $\mathrm{Det}(g)$, is equal to 1. Orthogonal transformations for which $\mathrm{Det}(g) = -1$ are called *improper* orthogonal transformations. As an example of an improper orthogonal transformation, consider reflection in the origin. This transformation has a matrix of the form

$$g_- = \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}.$$

If g is an arbitrary improper orthogonal transformation, that is, if $\mathrm{Det}(g) = -1$, then gg_- is a transformation for which $\mathrm{Det}(gg_-) = \mathrm{Det}(g) \cdot \mathrm{Det}(g_-) = 1$, and hence gg_- is a rotation. Therefore $g = (gg_-)g_-$ is the product of a rotation and reflection in the origin.

2. Introduction of parameters in the rotation group. In the sequel, we shall require several means of describing rotations in terms of parameters. Since every rotation is a rotation around some axis, we define the rotation completely by specifying the axis of rotation and the angle through which the rotation is made. We shall often describe a rotation by the vector $\xi = (\xi_1, \xi_2, \xi_3)$, where the direction of ξ lies on the axis of rotation and the length of ξ is equal to the angle of rotation about the axis. We choose that direction for the vector ξ such that looking from its end, the angle of rotation does not exceed π. In this fashion, the co-ordinates of the vectors which describe all possible rotations satisfy the condition $\xi_1^2 + \xi_2^2 + \xi_3^2 \leq \pi$, and they clearly are the entire solid sphere of radius π. It is clear that distinct interior points of this sphere correspond to distinct rotations, while diametrically opposite points on the surface of the sphere correspond to one and the same rotation through the angle π (since rotations through the angle π in two opposite directions lead to the same result).

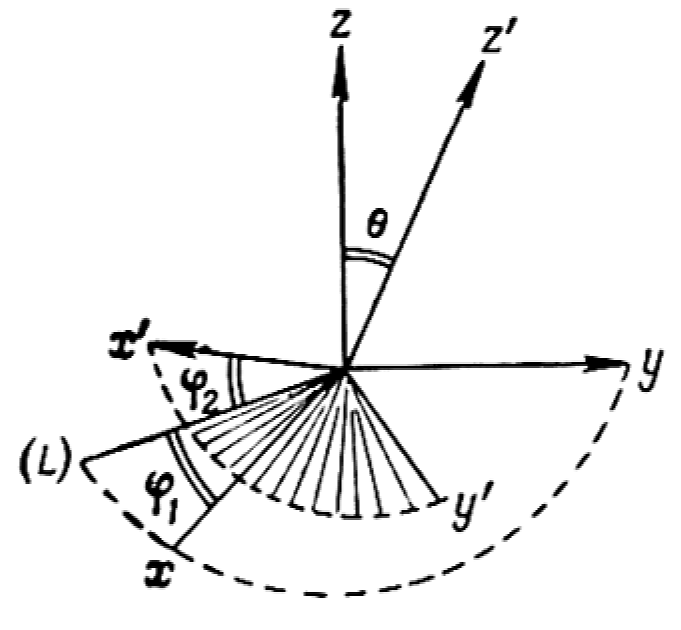

Fig. 1.

The above method of describing rotations shows the topological structure of the rotation group. In fact, this group is homeomorphic to the solid sphere in which one identifies diametrically opposite points of the boundary.

An important method of parametrizing the rotation group is provided by what are called the Eulerian angles. Let the co-ordinate axes ox, oy, and oz be carried by the rotation g into the lines ox', oy', and oz' respectively. Let (L) denote the line of intersection of the planes xoy and $x'oy'$. Let ϕ_1 denote the angle between the lines ox and (L). Let ϕ_2 denote the angle between the lines (L) and ox'. Let θ denote the angle between the positive directions of the axes oz and oz'. It is clear that the rotation g can be represented as the product of three successive rotations, as follows. First, a rotation g_{θ_1} through the positive angle ϕ_1 around the axis oz; as a result of this, the axis ox is transformed into the line L. Then one makes a rotation g_θ through the angle θ around the line L. This carries the axis oz into the axis oz'. Finally, one makes a rotation g_{θ_2} through the angle θ_2 around the line oz'. Thus we have

$$g(\varphi_1, \theta, \varphi_2) = g_{\varphi_2} g_\theta g_{\varphi_1}. \tag{8}$$

We now determine the elements g_{ik} of the matrix of the rotation $g(\phi_1, \theta, \phi_2)$

whose Euler angles are ϕ_1, θ, ϕ_2. The matrices of the rotations $g_{\phi_1}, g_\theta, g_{\phi_2}$ around the axes oz, ox, and oz' have the form *

$$g_{\varphi_1} = \begin{Vmatrix} \cos\varphi_1 & -\sin\varphi_1 & 0 \\ \sin\varphi_1 & \cos\varphi_1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad g_\theta = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{Vmatrix}, \quad g_{\varphi_2} = \begin{Vmatrix} \cos\varphi_2 & -\sin\varphi_2 & 0 \\ \sin\varphi_2 & \cos\varphi_2 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

Upon carrying out these rotations one after the other, their matrices are multiplied, and hence we have

$$g(\varphi_1, \theta, \varphi_2) = g_{\varphi_2} g_\theta g_{\varphi_1} =$$

$$= \begin{Vmatrix} \cos\varphi_2\cos\varphi_1 - \cos\theta\sin\varphi_2\sin\varphi_1 & -\cos\varphi_2\sin\varphi_1 - \cos\theta\sin\varphi_2\cos\varphi_1 & \sin\varphi_2\sin\theta \\ \sin\varphi_2\cos\varphi_1 + \cos\theta\cos\varphi_2\sin\varphi_1 & -\sin\varphi_2\sin\varphi_1 + \cos\theta\cos\varphi_2\cos\varphi_1 & -\cos\varphi_2\sin\theta \\ \sin\varphi_1\sin\theta & \cos\varphi_1\sin\theta & \cos\theta. \end{Vmatrix} \tag{9}$$

The angles ϕ_1 and ϕ_2 vary from 0 to 2π, and the angle θ varies from 0 to π. Distinct triples of numbers, varying within these limits, correspond to distinct rotations, except in the cases $\theta=0$ and $\theta=\pi$. For $\theta=0$, our rotation is a rotation about the axis oz through the angle $\phi_1+\phi_2$, and for $\theta=\pi$, it is a rotation about the axis oz through the angle $\phi_1-\phi_2$. Thus, in these cases, distinct pairs of numbers (ϕ_1, ϕ_2) may correspond to the same rotation.

Consider again the rotation $g(\phi_1, \theta, \phi_2)$, which is completely determined by its matrix (9). Let us show that the inverse transformation is described by the Eulerian angles $\pi-\phi_2, \theta$, and $\pi-\phi_1$. In fact, if ϕ_1 is replaced by $\pi-\phi_2$ and ϕ_2 is replaced by $\pi-\phi_1$ in the matrix (9), then the matrix is changed into its transpose, and since (9) is an orthogonal matrix, its inverse is equal to its transpose (see (5)). Therefore, *if the rotation* g *has Eulerian angles* ϕ_1, θ, ϕ_2, *then the rotation* g^{-1} *has Eulerian angles* $\pi-\phi_2$, θ, $\pi-\phi_1$.

3. Invariant integration. In studying functions defined on the group G (that is, functions of ϕ_1, θ, ϕ_2), we are led to consider integrals of these functions over the group (that is, over all admissible values of ϕ_1, θ, ϕ_2: $0 \le \phi_1 < 2\pi$, $0 \le \theta \le \pi$, $0 \le \phi_2 < 2\pi$). The most valuable for the theory of representations is what is called invariant integration. An invariant integral for functions $f(g) = f(\phi_1, \theta, \phi_2)$ is an integral of the form **

$$\int f(g)\,dg = \int f(\varphi_1, \theta, \varphi_2)\, I(\varphi_1, \theta, \varphi_2)\, d\varphi_1\, d\theta\, d\varphi_2$$

with respect to ϕ_1, θ, ϕ_2, with a "weight function" $I(\phi_1, \theta, \phi_2)$ chosen in such a fashion that the equality

$$\int f(gg_0)\,dg = \int f(g)\,dg \tag{10}$$

---

* Translator's note. The authors have reversed the correct order of writing the product of the matrices g_{ϕ_1}, g_θ, and g_{ϕ_2}. Actually, the matrix of the rotation g with Euler angles ϕ_1, θ, ϕ_2 is equal to $g_{\phi_1} g_\theta g_{\phi_2}$, where the matrices g_{ϕ_1}, g_θ and g_{ϕ_2} are defined as above. This formula reverses the usual order of writing matrix products, and it does so because the matrices g_θ and g_{ϕ_2} are not given in the original co-ordinate system but in "running" co-ordinates. It follows that the matrix (9) can be made correct by interchinging ϕ_1 and ϕ_2 throughout.

** We shall sometimes write $dg \equiv I\,d\varphi_1\,d\theta\,d\varphi_2$.

holds. Thus the invariant integral of a function $f(g)$ does not change if the argument g is replaced by gg_0, that is, if the function $f(g)$ is translated. It can be proved that the weight function is uniquely defined, up to a constant factor, by the condition (10) of invariance. We choose this factor so that the integral of the function $f(g) \equiv 1$ is equal to 1, that is,

$$\int dg = 1.$$

In defining the invariant integral, we could use any parameters at all that define the rotation. However, the Eulerian angles are the most convenient of all for writing down the weight function.

Consider a rotation g with Eulerian angles ϕ_1, θ, ϕ_2. Let P be the point on the unit sphere which goes into the north pole of the sphere under the rotation g. Let Q be the point on the unit sphere which lies on the axis ox after being subjected to the rotation g, It is clear that the rotation is completely determined by specifying the points P and Q. The point P can of course be any point whatsoever on the unit sphere, while the point Q lies on a great circle whose plane is perpendicular to the radius OP.

The elements of the third row of the matrix (9) are the Cartesian co-ordinates of the point P. It is clear from this that the spherical co-ordinates of the point P are $\pi/2 - \phi_2$ and θ.

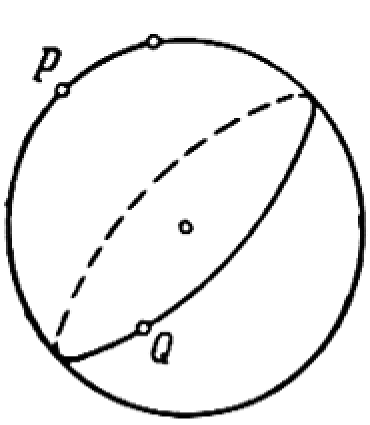

Fig 2.

If g_0 is another rotation and $\tilde{g} = gg_0$, then the points $\tilde{P}$ and $\tilde{Q}$ which correspond to the rotation $\tilde{g}$ are obtained from the points P and Q by subjecting P and Q to the rotation g_0.

Let us show that the invariant integral is given by the formula

$$\int f(g)\, dg = \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} f(\varphi_1, \theta, \varphi_2) \sin\theta \, d\varphi_1 \, d\theta \, d\varphi_2,$$

that is, let us show that under this definition of the integral, the relation

$$\int f(g)\, dg = \int f(gg_0)\, dg$$

holds for all g_0. Let $\tilde{\phi}_1$, $\tilde{\theta}$, and $\tilde{\phi}_2$ denote the Eulerian angles of the rotation g. We shall show that the formula

$$\sin\theta \, d\varphi_1 \;\; d\theta \;\; d\varphi_2 = \sin\tilde{\theta} \, d\tilde{\varphi}_1 \, d\tilde{\theta} \, d\tilde{\varphi}_2.$$

obtains, or, symbolically,

$$dg = d\tilde{g}.$$

The expression $\sin\theta d\phi_1 d\theta d\phi_2$ has a simple geometric meaning. In fact, $\sin\theta d\phi_1 d\theta$ is the element of area of the surface of the unit sphere at the point P. The differential $d\phi_2$ is the element of arc of the great circle on which Q lies. In fact, if the angle ϕ_2 is changed by an amount $d\phi_2$ while ϕ_1 and θ are held fixed (that is, while the point P is held fixed), this leads to an additional rotation of amount $d\phi_2$ around the point P, that is, a movement of the point Q through the angle $d\phi_2$. As was shown above, substitution of $\tilde{g} = gg_0$ for g leads to a rotation of the pair, P,

Q by the rotation g_0. But under rotation, neither the element of area nor the element of arc change. Hence we have

$$\sin\theta\, d\varphi_1\, d\theta\, d\varphi_2 = \sin\tilde{\theta}\, d\tilde{\varphi}_1\, d\tilde{\theta}\, d\tilde{\varphi}_2.$$

We must prove the equality

$$\int f(\tilde{g})\, dg = \int f(g)\, dg,$$

or, equivalently,

$$\int f(\tilde{g})\, dg = \int f(\tilde{g})\, d\tilde{g}.$$

This last follows from the fact that $d\tilde{g} = dg$, as we have already proved.

If we demand in addition that the integral be normalized, that is, that $\int dg = 1$, then, since $\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \sin\theta d\phi_1 d\theta d\phi_2 = 8\pi^2$, we must set

$$\int f(g)\, dg = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f(\varphi_1, \theta, \varphi_2)\sin\theta\, d\varphi_1\, d\theta\, d\varphi_2. \tag{11}$$

4. Connection between the rotation group and the group of unitary matrices of order 2. In this section, we show that rotations of 3-dimensional space can be described by means of complex matrices of order 2. For this purpose, we consider stereographic projection of a sphere onto a plane. As is known, this consists in mapping each point P of the sphere into the point M in the plane in question which lies on the line $O'P$ (here O' denotes the north pole of the sphere). The relation of the plane to the sphere is illustrated in Figure 3. Every rotation of 3-dimensional space around the center of the sphere carries the sphere onto itself and hence engenders a certain transformation in the plane. Our immediate problem is to examine these transformations more closely.

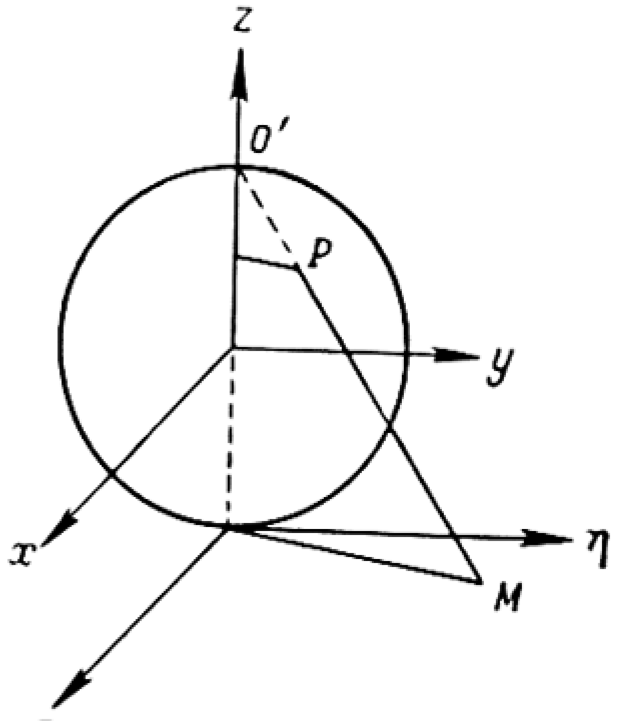

Fig. 3.

We consider a sphere of diameter 1. By comparing similar triangles (see Figure 3), it is easy to establish the relations existing between the co-ordinates x, y, and z of the point P on the sphere and the co-ordinates ξ, η of the point M in the plane: $\xi = x/\tfrac{1}{2} - z$, $\eta = y/\tfrac{1}{2} - z$. It is convenient to introduce the complex variables $\zeta = \xi + i\eta$. Then

$$\zeta = \xi + i\eta = \frac{x + iy}{\frac{1}{2} - z}. \tag{12}$$

Since we have $x^2 + y^2 = \frac{1}{4} - z^2$ on the sphere, it follows that

$$\zeta = \frac{x^2 + y^2}{\left(\frac{1}{2} - z\right)(x - iy)} = \frac{\frac{1}{2} + z}{x - iy}. \tag{12'}$$

We now find the transformation of the plane which corresponds to a rotation through the angle ϕ around the axis oz. We have

$$x' = x\cos\varphi - y\sin\varphi,$$
$$y' = x\sin\varphi + y\cos\varphi,$$
$$z' = z.$$

From this it follows that

$$\zeta' = \frac{x'+iy'}{\frac{1}{2}-z} = \frac{e^{i\varphi}(x+iy)}{\frac{1}{2}-z} = e^{i\varphi}\zeta.$$

That is, the transformation of the plane corresponding to this rotation is

$$\zeta' = e^{i\phi}\zeta. \tag{13}$$

We consider now a rotation through the angle θ around the axis ox. As in the above case, the expression

$$w = \frac{y+iz}{\frac{1}{2}-x}$$

is multiplied by $e^{i\theta}$; that is,

$$w' = e^{i\theta}w. \tag{14}$$

It remains to compute w in terms of ζ (and correspondingly, w' in terms of ζ'). We consider the expression

$$\frac{w+i}{w-i} = \frac{\frac{y+iz}{\frac{1}{2}-x}+i}{\frac{y+iz}{\frac{1}{2}-x}-i} = \frac{-(x+iy)+\left(\frac{1}{2}+z\right)}{(x-iy)-\left(\frac{1}{2}-z\right)}.$$

Using formulas (12) and (12′), we obtain

$$\frac{w+i}{w-i} = \frac{-\zeta\left(\frac{1}{2}-z\right)+\left(\frac{1}{2}+z\right)}{\frac{1}{\zeta}\left(\frac{1}{2}+z\right)-\left(\frac{1}{2}-z\right)} = \zeta$$

and analogously

$$\frac{w'+i}{w'-i} = \zeta'.$$

From this, we find by expressing w and w' in terms of ζ and ζ' and then substituting in formula (14) that

$$\frac{\zeta'+1}{\zeta'-1} = e^{i\theta}\frac{\zeta+1}{\zeta-1}.$$

Solving this equation for ζ', we obtain the transformation of the plane which corresponds to rotation around the axis ox through the angle θ:

$$\zeta' = \frac{\zeta(e^{i\theta}+1)+(e^{i\theta}-1)}{\zeta(e^{i\theta}-1)+(e^{i\theta}+1)} = \frac{\zeta\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{i\zeta\sin\frac{\theta}{2}+\cos\frac{\theta}{2}}. \tag{15}$$

We see in this way that rotations around the axes ox and oz correspond to fractional linear transformations in the ζ-plane. It is also clear that products of rotations correspond to products of the corresponding transformations in the plane. Since every rotation can be written as the product of rotations around the axes ox and oz (see para. 2), it follows that with every rotation of 3-dimensional space we can associate a product of transformations of the form (13) and (15), that is, a certain fractional linear transformation

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}. \tag{16}$$

The fractional linear transformation (16) is completely determined by the complex matrix of order 2

$$\begin{Vmatrix} \alpha & \beta \\ \gamma & \delta \end{Vmatrix}. \tag{17}$$

Since the functional value ζ' in formula (16) does not change if both numerator and denominator of the right-hand side are multiplied by the same non-zero complex number, we may multiply the numbers α, β, γ, and δ by $\pm \frac{1}{\sqrt{\alpha\delta - \beta\gamma}}$ and thus make the determinant of the matrix (17) equal to 1.*

In this way, we associate with every rotation of 3-dimensional space a matrix of the form (17), determined except for the sign of its entries, for which $\alpha\delta - \beta\gamma = 1$. We set down, in particular, the matrices which correspond to rotation g_ϕ and g_θ around the axes oz and ox respectively. Corresponding to the rotation g_θ, we have the matrix

$$g_\theta \sim \begin{Vmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{Vmatrix}. \tag{18}$$

The transformation $\zeta' = e^{i\phi}\zeta$ corresponds to the rotation g_ϕ. Writing this in the form

$$\zeta' = \frac{e^{i\frac{\varphi}{2}}\zeta}{e^{-i\frac{\varphi}{2}}},$$

we obtain the matrix of the transformation

$$g_\varphi \sim \begin{Vmatrix} e^{i\frac{\varphi}{2}} & 0 \\ 0 & e^{-i\frac{\varphi}{2}} \end{Vmatrix} \tag{19}$$

with determinant equal to 1.

The rotation g with Eulerian angles ϕ_1, θ, ϕ_2 can be written in the form $g = g_{\phi_1} g_\theta g_{\phi_2}$. Since the matrices of fractional linear transformations multiply when the corresponding transformations are carried out in succession, it follows that the matrix corresponding to the rotation g is given by

$$g \sim \begin{Vmatrix} e^{i\frac{\varphi_2}{2}} & 0 \\ 0 & e^{-i\frac{\varphi_2}{2}} \end{Vmatrix} \begin{Vmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{Vmatrix} \begin{Vmatrix} e^{i\frac{\varphi_1}{2}} & 0 \\ 0 & e^{-\frac{i\varphi_1}{2}} \end{Vmatrix} =$$

$$= \begin{Vmatrix} \cos\frac{\theta}{2} e^{i\frac{\varphi_1+\varphi_2}{2}} & i\sin\frac{\theta}{2} e^{i\frac{\varphi_2-\varphi_1}{2}} \\ i\sin\frac{\theta}{2} e^{-i\frac{\varphi_2-\varphi_1}{2}} & \cos\frac{\theta}{2} e^{-i\frac{\varphi_1+\varphi_2}{2}} \end{Vmatrix}. \tag{20}$$

The matrices (18) and (19) are unitary matrices with determinant 1. Therefore the matrix (20), which is the product of such matrices, and which corresponds to an arbitrary rotation, is also a unitary matrix with determinant 1.

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* This condition determines the coefficients of a fractional linear transformation except for their sign. We shall discuss the sign in more detail in the sequel.

We shall now prove the converse: to every unitary matrix with determinant 1, there corresponds a certain rotation. From the definition of a unitary matrix, we have

$$\alpha\bar{\gamma}+\beta\bar{\delta}=0,\quad \alpha\bar{\alpha}+\beta\bar{\beta}=1,\quad \gamma\bar{\gamma}+\delta\bar{\delta}=1.$$

From these relations and the condition $\alpha\delta-\beta\gamma=1$, we easily infer that $\delta=\bar{\alpha}$, $\gamma=-\bar{\beta}$. Hence an arbitrary matrix of the kind under consideration can be represented in the form

$$\begin{Vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{Vmatrix}, \tag{21}$$

where

$$|\alpha|^2+|\beta|^2=1. \tag{21'}$$

It is plain that the matrix (21) with the additional condition (21′) can be represented in the form (20). It is necessary only to set

$$\cos\frac{\theta}{2}=|\alpha|,\quad \sin\frac{\theta}{2}=|\beta|,\ ^{*}$$

and to determine the angles ϕ_1 and ϕ_2 by the equations

$$\frac{\varphi_1+\varphi_2}{2}=\arg\alpha,$$

$$\frac{\varphi_2-\varphi_1}{2}+\frac{\pi}{2}=\arg\beta.$$

In this fashion, every rotation can be described by two complex numbers satisfying condition (21′), or, equivalently, by four real numbers, the sum of whose squares is equal to 1.

We have accordingly shown that to every unitary matrix of order 2 with determinant 1 there corresponds a rotation in 3-dimensional space. Conversely, as was shown earlier, to every such rotation, there correspond two such matrices, which differ in sign.

The correspondence established above between rotations and fractional linear transformations is one-to-one. On the other hand, we have seen that every fractional linear transformation can be written with the help of two distinct matrices of determinant 1. Thus, to every rotation there correspond two matrices of the form (21). It would be foolish to suppose that we can rid ourselves in some natural way of the two-valuedness of this correspondence by choosing the sign in some determined way. Consider, for example, the rotation through an angle ϕ around the axis oz. The corresponding matrix is

$$\begin{Vmatrix} e^{i\frac{\varphi}{2}} & 0 \\ 0 & e^{-i\frac{\varphi}{2}} \end{Vmatrix}.$$

In particular, two matrices correspond to the identity rotation ($\phi=2k\pi$):

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}. \tag{22}$$

If we picked only the matrix

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[*] We recall that $0\leq\theta\leq\pi$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

to correspond to this rotation, then, as the angle ϕ moves continuously from 0 to 2π, we should obtain for $\phi = 2\pi$, the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, if we do not wish to destroy continuity, we must suppose that both of the matrices (22) correspond to the identity rotation.

We write the matrix of the rotation g in terms of the parameters α and β:

$$\| g_{ik} \| = \begin{Vmatrix} \frac{1}{2}(\alpha^2 - \beta^2 + \bar{\alpha}^2 - \bar{\beta}^2) & \frac{i}{2}(-\alpha^2 - \beta^2 + \bar{\alpha}^2 + \bar{\beta}^2) & -\alpha\beta + \bar{\alpha}\beta \\ \frac{i}{2}(\alpha^2 - \beta^2 - \bar{\alpha}^2 + \bar{\beta}^2) & \frac{i}{2}(\alpha^2 + \beta^2 + \bar{\alpha}^2 + \bar{\beta}^2) & -i(\alpha\beta - \bar{\alpha}\bar{\beta}) \\ \alpha\bar{\beta} + \bar{\alpha}\beta & i(-\alpha\bar{\beta} + \bar{\alpha}\beta) & \alpha\bar{\alpha} - \beta\bar{\beta} \end{Vmatrix}. \tag{23}$$

To prove this formula, substitute the elements of the matrix (20) for α and β in (23). After some simple reductions, we arrive at the expression (9) for the matrix in terms of the Eulerian angles.

Setting $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ and recalling that $|\alpha|^2 + |\beta|^2 = 1$, we see that the group of rotations of 3-dimensional space is mapped into the sphere

$$\alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2 = 1 \tag{24}$$

lying in 4-dimensional space. Note that one and the same rotation correspond to each pair of diametrically opposite points on the sphere.

5. Definition of representations of the rotation group. We shall give in this paragraph the principal definitions which will be used in the sequel. We shall say that we have a finite-dimensional representation of a group, $g \to T_g$, if for every group element g, there exists a linear transformation T_g in a certain linear space R* with the properties that the product of two group elements corresponds to the product of the linear transformations and such that the group identity corresponds to the identity transformation. That is,

$$T_{g_1} T_{g_2} = T_{g_1 g_2} \tag{25}$$

and

$$T_e = E. \tag{26}$$

Since every linear transformation of a finite-dimensional space can be described by a matrix, we may also define a finite-dimensional representation as a correspondence which associates with every group element g a matrix T_g in such a way that relations (25) and (26) hold.

A representation $g \to T_g$ of the rotation group is said to be continuous if the elements of the matrix T_g depend continuously on g. In the sequal we shall consider only continuous representations. A trivial example of a representation is the correspondence which associates with every group element the identity matrix. Such a representation of the rotation group is said to be an *identity* representation. Another example of a representation of the rotation group is the *principal* representation. This representation associates with every group element its matrix in some basis.

---

* In the sequal, we shall also refer to elements of the space R as quantities transformed under the representation $g \to T_g$.

A subspace R_1 of the space R in which the linear transformations of a representation $g \to T_g$ operate is said to be *invariant with respect to this representation* if, for all ξ in R_1 and g in G, the vector $T_g \xi$ is also in R_1. That is, the subspace R_1 is invariant under each of the transformations T_g.

If the only subspaces of the space R which are invariant under the representation $g \to T_g$ are the entire space R and the subspace consisting of 0 alone, then the representation is said to be *irreducible*. We shall see a little later that the study of an arbitrary representation of the rotation group can be reduced to the study of irreducible representations. We shall classify irreducible representations in §2, and the matrices T_g of irreducible representations will be described in §7. Various realizations of representations of the rotation group (that is, various realizations of the space R and the transformations T_g) will be studied in §§3, 5, 6, and 8.

A representation $g \to T_g$ is said to be *unitary* if there exists a scalar product in the complex space R with respect to which all of the transformations T_g are unitary. We shall show *that every representation of the rotation group is unitary, that is, that one can introduce a scalar product in such a way that all of the transformations T_g are unitary with respect to this scalar product.* In order to do this, we consider an arbitrary scalar product (ξ, η) in R (here ξ and η denote arbitrary elements of R). In general, T_g will not be unitary with respect to this scalar product. That is, $(T_g \xi, T_g \eta)$ may be different from (ξ, η). We *average* the function $(T_g \xi, T_g \eta)$ over the group. That is, we consider the expression

$$\int (T_g \xi,\ T_g \eta)\, dg,$$

where the integral is understood to be the invariant integral defined in paragraph 3.

We now define a new scalar product by the formula

$$(\xi,\ \eta)_1 = \int (T_g \xi,\ T_g \eta)\, dg.$$

We verify that $(\xi, \eta)_1$ enjoys all of the properties of a scalar product. In fact,

$$(\xi_1 + \xi_2,\ \eta)_1 = \int (T_g(\xi_1 + \xi_2),\ T_g \eta)\, dg =$$
$$= \int (T_g \xi_1,\ T_g \eta)\, dg + \int (T_g \xi_2,\ T_g \eta)\, dg = (\xi_1,\ \eta)_1 + (\xi_2,\ \eta)_1.$$

Analogously, one can show that $(\xi, \eta)_1 = \overline{(\eta, \xi)_1}$ and that $(\lambda\xi, \eta)_1 = \lambda(\xi, \eta)_1$. Furthermore, $(\xi, \xi)_1 = \int (T_g \xi, T_g \xi) dg > 0$ for $\xi \neq 0$, since $(T_g \xi, T_g \xi) > 0$ for all g.*

We show finally that all of the transformations T_g are unitary with respect to the scalar product $(\xi, \eta)_1$ that is, that $(T_{g_0}\xi, T_{g_0}\eta)_1 = (\xi, \eta)_1$ for all g_0 in the group and ξ, η in R. Write $T_{g_0}\xi = \xi'$ and $T_{g_0}\eta = \eta'$. Then

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* Translator's note. This observation of course makes tacit use of the fact that $\int f(g) dg > 0$ for all continuous functions which are positive everywhere. A little more is actually true of the invariant integral: $\int f(g) dg > 0$ if $f \geq 0$, f is continuous, and $f(g) > 0$ for some g in the group.

$$(T_{g_0}\xi,\ T_{g_0}\eta)_1 = (\xi',\ \eta')_1 = \int (T_g\xi',\ T_g\eta')\,dg = \int (T_gT_{g_0}\xi,\ T_gT_{g_0}\eta)\,dg =$$

$$= \int (T_{gg_0}\xi,\ T_{gg_0}\eta)\,dg.$$

Furthermore, in view of the invariance of the integral $(\int f(gg_0)dg = \int f(g)dg)$, we have

$$\int (T_{gg_0}\xi,\ T_{gg_0}\eta)\,dg = \int (T_g\xi,\ T_g\eta)\,dg = (\xi,\ \eta)_1.$$

This implies, finally, that

$$(T_{g_0}\xi,\ T_{g_0}\eta)_1 = (\xi,\ \eta)_1.$$

In this manner, we have shown that the representation $g \rightarrow T_g$ is unitary with respect to the scalar product $(\xi, \eta)_1$.

A study of unitary representations of the group can be reduced to a study of irreducible unitary representations. In fact, consider a unitary representation $g \rightarrow T_g$. If there are no proper (i.e. distinct from R and 0) invariant subspaces of R, then the representation is irreducible. On the other hand, if R contains a proper invariant subspace R_1, then the set of all vectors orthogonal to R_1 forms a subspace which is also invariant. The entire space R thus can be written as the direct sum of two mutually orthogonal subspaces R_1 and R_2, both of which are invariant. If the representation is reducible in either R_1 or R_2, then we continue the reduction process, until we come to irreducible representations.

So far, we have considered only finite dimensional representations of the rotation group. However, we shall also encounter infinite dimensional unitary representations (for example, in §3, para. 5 and §7, para. 2). We shall say that we have an infinite dimensional unitary representation if, to every group element g there corresponds a unitary transformation T_g in some Hilbert space (that is, an infinite dimensional Euclidean space) having the properties that (25) and (26) are satisfied. The representation is said to be continuous, if all vectors ξ and η, $(T_g\xi, \eta)$ is a continuous function of g.

The theorem on the decomposability of representations into irreducible components is valid also for infinite-dimensional representations of the rotations group. We state it without proof. *Let there be given a unitary representation* $g \rightarrow T_g$ *of the rotation group in the Hilbert space* R. *Then there exist finite-dimensional subspaces* $R_1, R_2, \cdots, R_n, \cdots$ *invariant with respect to* T_g, *in each of which the representation* T_g *is irreducible. These subspaces* R_i *are pairwise orthogonal, and the sum of the* R_i *is the entire space* R. *This means that every vector* ξ *in* R *can be written as a convergent series* *

$$\xi = \xi_1 + \xi_2 + \ldots + \xi_n + \ldots,$$

where the vectors ξ_i *belong to the invariant subspaces* R_i. **

---

* The series $\xi_1 + \xi_2 + \cdots + \xi_n + \cdots$ is said to converge to ξ if $s_n = \xi_1 + \cdots + \xi_n$ converges to ξ, that is, if $(\xi - S_n, \xi - S_n) \rightarrow 0$ as $n \rightarrow \infty$.

** Translator's note. For this theorem to hold, the Hilbert space R must obviously be separable.

§2. Infinitesimal rotations and description of the irreducible representations of the rotation group

In this section, we shall describe all of the representations of the rotation group. We already know that every representation is equivalent to a unitary representation, and hence we may limit ourselves to identifying the unitary representations of this group. We shall first find all of the irreducible representations, and then we shall show how an arbitrary representation can be decomposed into irreducible parts.

1. Definition of the matrices A_k corresponding to infinitesimal rotations. Let there be given a unity representation of the rotation group G. As we know, this means that to every rotation g, there corresponds a certain unitary matrix $T_g = \| a_{ik}(g) \|$ such that to the product of the rotations g_1 and g_2 there corresponds the product of the matrices T_{g_1} and T_{g_2}. That is,

$$T_{g_1 g_2} = T_{g_1} T_{g_2}. \tag{1}$$

In particular, the identity matrix E corresponds to the group identity $g = e$.

As parameters defining the rotation g, we take the co-ordinates ξ_1, ξ_2, ξ_3 of the vector directed along the axis of rotation whose length is equal to the angle of the rotation (these parameters were introduced in para. 2 of the preceding section). Then the matrix T_g is a function of these same parameters, that is, $T_g = T(\xi_1, \xi_2, \xi_3)$. One can show that T_g possesses derivatives of all orders with respect to the variables ξ_1, ξ_2, ξ_3.* Since the vector $\xi_1 = \xi_2 = \xi_3 = 0$ corresponds to the rotation through the angle 0, that is, to the identity rotation, it follows that

$$T(0, 0, 0) = E. \tag{2}$$

Let us expand $T(\xi_1, \xi_2, \xi_3)$ in a Taylor's series about the values $\xi_1 = \xi_2 = \xi_3 = 0$. This gives us

$$T(\xi_1, \xi_2, \xi_3) = E + A_1\xi_1 + A_2\xi_2 + A_3\xi_3 + \cdots, \tag{3}$$

where E is the identity matrix and A_1, A_2, A_3 are constant matrices, being the partial derivatives of the matrix $T(\xi_1, \xi_2, \xi_3)$ with respect to ξ_1, ξ_2, ξ_3, respectively, computed at the point $\xi_1 = \xi_2 = \xi_3 = 0$. The symbol " ... " means that we have not written the remainder term of Taylor's series, which is small of an order higher than the first with respect to $\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$.

We shall show that the representation, that is, the function $T(\xi_1, \xi_2, \xi_3)$, is completely determined by the matrices A_1, A_2, A_3, and furthermore we shall find all possible triples of matrices A_1, A_2, A_3.

The matrices A_1, A_2, A_3 have a simple interpretation. To explain this, we consider a rotation through the angle ξ_1 around the axis ox. In the representation, the matrix corresponding to this rotation is

$$T(\xi_1, 0, 0) = E + A_1\xi_1 + \cdots.$$

---

* For the proof, see the appendix to the present section. We recall that dependence of a matrix on certain variables and differentiability of the matrix with respect to these variables means simply that the elements of the matrix are functions of these variables and differentiable with respect to them.

It is clear from this formula that the matrix $T(\xi_1, 0, 0)$, corresponding to a rotation through an infinitesimal angle ξ_1 around the axis ox, is defined to within infinitesimals of higher order by the matrix A_1. The matrices A_1, A_2, A_3 are called *the matrices which correspond to infinitesimal rotations about the co-ordinate axes.*

We shall now show that the matrices A_k define the representation, that is, that if we know only these three matrices, we can find $T(\xi_1, \xi_2, \xi_3)$ for arbitrary ξ_1, ξ_2, ξ_3. To do this, we select an arbitrary vector (ξ_1, ξ_2, ξ_3) and consider two rotations around this vector: $g(t\xi_1, t\xi_2, t\xi_3)$ and $g(s\xi_1, s\xi_2, s\xi_3)$.* The product of these two rotations is evidently a rotation around the same axis defined by the parameters $(t+s)\xi_1$, $(t+s)\xi_2$, $(t+s)\xi_3$:

$$g((t+s)\xi_1,\ (t+s)\xi_2,\ (t+s)\xi_3) = g(t\xi_1,\ t\xi_2,\ t\xi_3)\, g(s\xi_1,\ s\xi_2,\ s\xi_3). \tag{4}$$

Since products are preserved under the representation $g \to T_g$, we have also

$$T((t+s)\xi_1,\ (t+s)\xi_2,\ (t+s)\xi_3) = T(t\xi_1,\ t\xi_2,\ t\xi_3)\, T(s\xi_1,\ s\xi_2,\ s\xi_3). \tag{4'}$$

We now differentiate both sides of (4') with respect to s and set $s=0$. We obtain

$$\frac{d}{dt} T(t\xi_1,\ t\xi_2,\ t\xi_3) = \frac{d}{ds} T(s\xi_1,\ s\xi_2,\ s\xi_3)\Big|_{s=0} \cdot T(t\xi_1,\ t\xi_2,\ t\xi_3).$$

But in view of formula (3), we also have

$$\frac{d}{ds} T(s\xi_1,\ s\xi_2,\ s\xi_3)\Big|_{s=0} = A_1\xi_1 + A_2\xi_2 + A_3\xi_3.$$

It follows from this that the matrix $X(t) = T(t\xi_1, t\xi_2, t\xi_3)$ satisfies the differential equation

$$\frac{d}{dt} X(t) = (A_1\xi_1 + A_2\xi_2 + A_3\xi_3)\, X(t).^{**} \tag{5}$$

Furthermore, the initial conditions

$$X(0) = T(0,\ 0,\ 0) = E \tag{5'}$$

must be satisfied. From the uniqueness theorem for solutions of differential equations, it follows that the equation (5) and the initial conditions (5') determine $X(t)$ completely. In particular, $X(1) = T(\xi_1, \xi_2, \xi_3)$ is determined completely.

In this way, we have shown that a representation is completely defined by the matrices A_1, A_2, A_3 which correspond to infinitesimal rotations.

We can give an explicit solution for equation (5). Its solution which satisfies the initial conditions (5') has the form

$$X(t) = e^{t(A_1\xi_1 + A_2\xi_2 + A_3\xi_3)}.^{***}$$

In particular,

$$T(\xi_1,\ \xi_2,\ \xi_3) = e^{A_1\xi_1 + A_2\xi_2 + A_3\xi_3}. \tag{6}$$

Thus, if the matrices A_1, A_2, A_3 correspond under the representation to infinitesimal rotations around the axes of co-ordinates, then the matrices $T_g = T(\xi_1, \xi_2, \xi_3)$ which give this representation are defined from the A_k by means of formula (6).

---

* These rotations are rotations through the angles $t\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$ and $s\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$, respectively.

** Equality (5) represents the system of linear differential equations which is obtained by equating the elements of the matrices which appear on the left and right sides.

*** See the appendix to §2.

2. Relations among the matrices A_k. We now explain what relations must exist among the matrices A_k in order that the matrices T_g defined by formula (6) should actually produce a unitary representation of the rotation group. With this end in view, we first develop the equations which the matrices A_k must satisfy, and then in paragraph 3, we solve these equations.

Let g_0 be a fixed rotation and g an arbitrary one. We consider the rotation

$$\tilde{g}_0 = g g_0 g^{-1}.$$

The matrices of the rotations $\tilde{g}_0$ and g_0 are obtained from each other by the same transformation. This implies that both of these rotations are rotations through one and the same angle ϕ, each about its own axis. If the rotation g_0 is defined by the vector $\eta = (\eta_1, \eta_2, \eta_3)$, then $\tilde{g}_0$ is defined by the vector $\tilde{\eta}$, The image of η under the rotation g:

$$\tilde{\eta} = g\eta.$$

In fact, the vector η is invariant under the rotation g_0; that is, $g_0\eta = \eta$. Hence $\tilde{g}_0\tilde{\eta} = g g_0 g^{-1}\tilde{\eta} = \tilde{\eta}$. The last equality signifies that $\tilde{\eta}$ is invariant under the rotation $\tilde{g}_0$, that is, it lies on the axis of rotation. Since the absolute value of the vector η, equal to the angle of rotation, does not alter under the passage from η to $g\eta$, it follows that $|\tilde{\eta}| = |\eta|$ and $\tilde{\eta}$ is the vector which defines the rotation $\tilde{g}_0$.

Since the product of rotations corresponds to the product of matrices T_g, in a representation, we infer from the equality $\tilde{g}_0 = g g_0 g^{-1}$ the equality

$$T_g T_{g_0} T_{g^{-1}} = T_{\tilde{g}_0}. \tag{7}$$

We can now obtain the relations among the matrices A_1, A_2, A_3 from formula (7). For this, we suppose that the rotation g_0, and therefore $\tilde{g}_0$, is small (i.e., that the vectors η and $\tilde{\eta}$ are small vectors), and we represent T_{g_0} and $T_{\tilde{g}_0}$ by means of the formula (3):

$$T_{g_0} = E + A_1\eta_1 + A_2\eta_2 + A_3\eta_3 + \dots,$$
$$T_{\tilde{g}_0} = E + A_1\tilde{\eta}_1 + A_2\tilde{\eta}_2 + A_3\tilde{\eta}_3 + \dots.$$

Substituting these expressions in (7), we find

$$T_g(E + A_1\eta_1 + A_2\eta_2 + A_3\eta_3 + \dots)T_{g^{-1}} = E + A_1\tilde{\eta}_1 + A_2\tilde{\eta}_2 + A_3\tilde{\eta}_3 + \dots. \tag{8}$$

With every small vector η, we associate the transformation $A_\eta = A_1\eta_1 + A_2\eta_2 + A_3\eta_3$, which defines the rotation given by this vector up to infinitesimals of order higher than the first. Then, equating the first-order members on both sides of (8), we obtain the formula

$$T_g A_\eta T_{g^{-1}} = A_{\tilde{\eta}}, \tag{9}$$

where $\tilde{\eta} = g\eta$.

It is clear that multiplying the vector η by a number results in multiplying the matrices A_η and $A_{\tilde{\eta}}$ by the same number. Hence we may drop the hypothesis that η is a small vector, once we have derived formula (9).

In order to obtain relations between A_1, A_2, and A_3 from (9), we now suppose that the rotation g is the rotation around the axis ox through a small angle α, while the vector η is the unit vector along the axis oy : $\eta = (0, 1, 0)$. Then $\tilde{\eta}$ is the vector obtained from η by the rotation g, that is, $\tilde{\eta}$ is the unit vector lying in the plane yoz and forming the angle a with the axis oy. Up to infinitesimals of order higher than the first with respect to a, therefore, the components of the vector will be

$$\tilde{\eta}_1 = 0,$$
$$\tilde{\eta}_2 = 1,$$
$$\tilde{\eta}_3 = \alpha.$$

In this way, writing T_g according to formula (3), where $\xi_1 = a$ and $\xi_2 = \xi_3 = 0$, we obtain

$$T_g = E_1 + A_1\alpha + \dots, \quad T_{g^{-1}} = E_1 - A_1\alpha + \dots,$$

$$A_\eta = A_2,$$
$$A_{\tilde{\eta}} = A_2 + \alpha A_3.$$

Substituting these expressions in formula (9) and equating the coefficients of a, we obtain the formula

$$A_1A_2 - A_2A_1 = A_3.$$

The expression $AB - BA$ is called the commutator of the matrices A and B and is denoted by the symbol $[A, B]$. That is, $AB - BA = [A, B]$. The relation just obtained can therefore be written in the form

$$[A_1, \ A_2] = A_3.$$

In the same way, we obtain two other relations:

$$[A_2, \ A_3] = A_1, \qquad [A_3, \ A_1] = A_2.$$

We have therefore shown that *if* $g \to T_g$ *is an arbitrary representation of the rotation group, then the matrices* A_1, A_2, A_3 *corresponding under this representation to infinitesimal rotations around the co-ordinate axes satisfy the relations*

$$\begin{aligned} [A_1, \ A_2] &= A_3, \\ [A_2, \ A_3] &= A_1, \\ [A_3, \ A_1] &= A_2. \end{aligned} \tag{10}$$

In the sequel, these relations will be referred to as the relations of commutation.

The formulas (10) show that if the matrices A_1, A_2, A_3 are replaced by orthonormal basis vectors for 3-dimensional space, then the relations of commutation for these matrices actually coincide with the formulas for the vector product of the basis vectors. For a given A_1, A_2, A_3, we define a linear space of matrices A_ξ of the form $A_1\xi_1 + A_2\xi_2 + A_3\xi_3$, where $\xi = (\xi_1, \xi_2, \xi_3)$ is an arbitrary vector. Then formulas (10) easily give the relation which expresses the commutator of two matrices of this space as a linear combination of the matrices A_k, that is, as an element of the same space. In fact, it follows from (10) that

$$[A_1\xi_1 + A_2\xi_2 + A_3\xi_3, \ A_1\eta_1 + A_2\eta_2 + A_3\eta_3] = A_3(\xi_1\eta_2 - \xi_2\eta_1) + \\ + A_1(\xi_2\eta_3 - \xi_3\eta_2) + A_2(\xi_3\eta_1 - \xi_1\eta_3).$$

Introducing the vector ζ equal to the vector product of ξ and η : $\zeta = [\xi, \eta]$, we may accordingly formulate the following assertion. *If the matrices* A_1, A_2, A_3 *correspond to infinitesimal rotations around the co-ordinate axes in a certain rep-*

resentation, then for any two matrices of the form $A_\xi = A_1\xi_1 + A_2\xi_2 + A_3\xi_3$ *and* $A_\eta = A_1\eta_1 + A_2\eta_2 + A_3\eta_3$, *the relation*

$$[A_\xi,\ A_\eta] = A_1\zeta_1 + A_2\zeta_2 + A_3\zeta_3,$$

holds where the vector $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ *is the vector product of the vectors* ξ *and* η.

In deriving the equations (10), we made no use of the fact that the representation T_g is the unitary. Let us see what restrictions on A_k are imposed by the condition that the representation be unitary. Setting $\xi_2 = \xi_3 = 0$ in formula (3), we obtain

$$T(\xi_1,\ 0,\ 0) = E + \xi_1 A_1 + \dots. \tag{3'}$$

Since the matrix T is unitary,

$$T^*(\xi_1,\ 0,\ 0)\,T(\xi_1, 0, 0) = E.$$

Replacing T in this last equality by its value from (3'), we find

$$(E + \xi_1 A_1^* + \dots)(E + \xi_1 A_1 + \dots) = E.$$

Equating the coefficient of ξ_1 in the left hand side of this equality to 0, we see that $A_1 + A_1^* = 0$, that is, $A_1^* = -A_1$.

Setting $H_1 = iA_1$, we have

$$H_1^* = H_1,$$

that is, H_1 is an Hermitain matrix. In the same way (setting $H_k = iA_k$), we obtain Hermitian matrices H_2 and H_3.* The relations of commutation (10) give analogous relations for the matrices H_1, H_2, H_3:

$$\begin{aligned}[][H_1,\ H_2] &= iH_3,\\ [H_2,\ H_3] &= iH_1,\\ [H_3,\ H_1] &= iH_2.\end{aligned} \tag{10'}$$

Consequently, we have reduced the problem of finding all representations of the rotation group to the *problem of finding all possible triples of Hermitian matrices* H_k *which satisfy the commutation relations* (10').

3. The form of an irreducible representation. Instead of looking for the matrices H_1, H_2, H_3, it is more convenient to consider the following linear combinations of them:

$$\begin{aligned}H_+ &= H_1 + iH_2,\\ H_- &= H_1 - iH_2,\\ H_3 &= H_3.\end{aligned}$$

It is easy to compute the commutators of these three matrices:

$$[H_+,\ H_3] = [H_1 + iH_2,\ H_3] = [H_1,\ H_3] + i[H_2,\ H_3] = -iH_2 - H_1 = -H_+.$$

We compute $[H_-, H_3]$ and $[H_+, H_-]$ in the same way. As a result, we obtain the formulas

$$\begin{aligned}[][H_+,\ H_3] &= -H_+,\\ [H_-,\ H_3] &= H_-,\\ [H_+,\ H_-] &= 2H_3.\end{aligned} \tag{11}$$

Besides this, we have

$$H_+^* = (H_1 + iH_2)^* = H_1 - iH_2 = H_-. \tag{12}$$

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* We note that, conversely, if the matrices H_k are Hermitian, then T_g is unitary, since (by (6)),

$$T_g = e^{\,i(H_1\xi_1 + H_2\xi_2 + H_3\xi_3)}.$$

Thus, our problem is reduced to the problem of finding matrices H_+, H_- and H_3 satisfying the conditions (11) and (12). We shall compute these matrices in the basis which consists of the eigen vectors of the matrix H_3. In this basis, the matrices H_+, H_- and H_3 are especially simple. As a preliminary, we first prove the following lemma.

Lemma. *Let f be an eigenvector of the transformation H_3 with corresponding eigenvalue λ:*

$$H_3 f = \lambda f.$$

Then the vector $f_1 = H_+ f$ is either the zero vector or an eigenvector of H_3 with corresponding eigenvalue $\lambda + 1$. Analogously, the vector $f_2 = H_- f$ either the zero vector or an eigenvector of H_3 with corresponding eigenvalue $\lambda - 1$.

In fact,

$$H_3 f_1 = H_3 H_+ f = [H_3, H_+] f + H_+ H_3 f = H_+ f + H_+ \lambda f = (\lambda + 1) H_+ f = (\lambda + 1) f_1.$$

In the same way, we see that $H_3 f_2 = (\lambda - 1) f_2$.

We turn now to the determination of the matrices H_+, H_-, and H_3.

Since H_3 is an Hermitian matrix, its eigenvalues are real. Let l denote the largest eigenvalue of the matrix H_3 and let f_l be any normalized eigenvector corresponding to this eigenvalue:

$$H_3 f_l = l f_l; \quad (f_l, f_l) = 1.$$

If $H_- f_l \neq 0$, then we set

$$H_- f_l = \alpha_l f_{l-1}$$

where we choose the number $\alpha_l > 0$ so that $(f_{l-1}, f_{l-1}) = 1$. In view of the lemma, f_{l-1} is a normalized eigenvector of H_3 corresponding to the eigenvalue $l-1$. If $H_- f_{l-1} \neq 0$, then, in the same way, we construct f_{l-2}, setting

$$H_- f_{l-1} = \alpha_{l-1} f_{l-2}, \text{ where } \alpha_{l-2} > 0 \text{ and } (f_{l-2}, f_{l-2}) = 1.$$

We continue this process, constructing

$$H_- f_{l-2} = \alpha_{l-2} f_{l-3}$$

and so on.

As the lemma shows, the vectors $f_l, f_{l-1}, \cdots$ are eigenvectors of the matrix H_3, with corresponding eigenvalues $l, l-1, \cdots$. Since the matrix H_3 has only a finite number of distinct eigenvalues, it follows that the sequence

$$f_l, f_{l-1}, f_{l-2}, \cdots$$

must terminate with some value k. That is, we have $H_- f_k = 0$ for some k. In this fashion, we have obtained a system of pairwise orthogonal normalized eigenvectors of the transformation H_3:

$$H_3 f_m = m f_m. \tag{13}$$

Besides this, we have

$$H_- f_m = \alpha_m f_{m-1}. \tag{14}$$

So that the formula (14) may hold for the last of the vectors constructed ($m = k$), we set $\alpha_k = 0$.

We now describe how the transformation H_+ acts on the vectors f_m. In the first place, the lemma shows that $H_+ f_m$ is either 0 or an eigenvector of H_3 with corresponding eigenvalue $m+1$. Since l is the great eigenvalue of H_3, it follows that

$$H_+ f_l = 0.$$

Let us find $H_+ f_{l-1}$. We have

$$H_+ f_{l-1} = \frac{1}{\alpha_l} H_+ H_- f_l = \frac{1}{\alpha_l} [H_+, H_-] f_l + \frac{1}{\alpha_l} H_- H_+ f_l = \frac{2}{\alpha_l} H_3 f_l = \frac{2l}{\alpha_l} f_l,$$

that is, the vector $H_+ f_{l-1}$ is a multiple of f_l:

$$H_+ f_{l-1} = \beta_l f_l, \qquad \beta_l > 0.$$

We now show that $H_+ f_m$ is a multiple of f_{m+1}, that is, $H_+ f_m = \beta_{m+1} f_{m+1}$. Suppose that this relation has been established for the vectors $f_l, f_{l-1}, \cdots, f_{m+1}$. We prove the relation for the vector f_m by the following calculation:

$$H_+ f_m = \frac{1}{\alpha_{m+1}} H_+ H_- f_{m+1} = \frac{1}{\alpha_{m+1}} [H_+, H_-] f_{m+1} + \frac{1}{\alpha_{m+1}} H_- H_+ f_{m+1} =$$
$$= \frac{2}{\alpha_{m+1}} H_3 f_{m+1} + \frac{\beta_{m+2}}{\alpha_{m+1}} H_- f_{m+2}.$$

Applying the equalities (13) and (14), we obtain

$$H_+ f_m = \frac{2(m+1) + \alpha_{m+2} \beta_{m+2}}{\alpha_{m+1}} f_{m+1}.$$

Setting

$$\frac{2(m+1) + \alpha_{m+2} \beta_{m+2}}{\alpha_{m+1}} = \beta_{m+1}, \tag{15}$$

we can write

$$H_+ f_m = \beta_{m+1} f_{m+1}. \tag{16}$$

Since $H_+ f_l = 0$, this formula is valid for $m = l$, if we set $\beta_{l+1} = 0$.

To define the transformations H_+ and H_-, we must first compute the coeficients α_m and β_m. Since $H_+^* = H_-$, we have

$$(H_+ f_{m-1}, f_m) = (f_{m-1}, H_- f_m).$$

Using (14) and (15), we obtain

$$\beta_m (f_m, f_m) = \alpha_m (f_{m-1}, f_{m-1}).$$

Since the eigenvectors f_m are normalized, it follows that $\alpha_m = \beta_m$. Substituting α for β and m for $m+1$ in (15), we obtain the following formulation for (15):

$$\alpha_m^2 - \alpha_{m+1}^2 = 2m.$$

In order to find α_m^2, we add these relations from $m = l$ to an arbitrary value m. We obtain

$$\alpha_m^2 - \alpha_{l+1}^2 = 2l + 2(l-1) + 2(l-2) + \ldots + 2m.$$

Using the fact that $\beta_{l+1}^2 = 0$ and that consequently $\alpha_{l+1}^2 = 0$, we find

$$\alpha_m^2 = (l+m)(l-m+1). \tag{17}$$

The formula just established makes it possible for us to find the number of vectors f_m in the sequence $f_l, \cdots, f_k$. We set $\alpha_k = 0$, where f_k is the last of the vectors constructed ($H_- f_k = 0$). From formula (17), it follows that $k = -l$. Since the number m in our process decreases by 1 at each step, the difference $l-(-l) = 2l$ must be an integer. Consequently, l itself must be either an integer or half of an odd integer (in the sequel, we shall for the sake of brevity refer to such numbers as half integers). The number of vectors f_m is obviously equal to

$2l+1$ in both cases.

Up to this point, we have imposed no restrictions on the representation T_g, that is, it may be either irreducible or reducible. We now consider irreducible representations. This means that in the space R, in which the representation acts, there are no proper subspaces which are invariant with respect to all of the transformations T_g. It follows that there are no proper subspaces of R which are invariant under the matrices A_1, A_2, A_3, or equivalently, under the matrices H_+, H_-, H_3. In fact, formula (6) shows that any such subspace would also be invariant under every transformation T_g. We shall show that in this case the vectors

$$f_l, f_{l-1}, \cdots, f_{-l}$$

form a basis for the space R. In fact, since H_+, H_-, and H_3 carry the vectors f_m into other vectors of the same system, it is clear that the subspace spanned by the vectors $f_m (m=-l, \cdots, l)$ is invariant with respect to H_+, H_-, and H_3. Consequently, this subspace must be R itself, since the representation is irreducible.

We have found therefore, that for an arbitrary irreducible representation, the transformations H_+, H_-, and H_3 can be described in an orthogonal basis consisting of normalized eigenvectors of H_3, by the formulas

$$\begin{aligned} H_+ f_m &= \alpha_{m+1} f_{m+1}, \\ H_- f_m &= \alpha_m f_{m-1}, \\ H_3 f_m &= m f_m; \end{aligned} \tag{18}$$

here $m=-l, -l+1, \cdots, l$, and l is an integer or a half integer, and

$$a_m = \sqrt{(l+m)(l-m+1)}.$$

The basis $f_l, f_{l-1}, \cdots, f_{-l}$ just described (for the case of an irreducible representation) will be called a canonical basis for this representation.

Returning to the transformations A_1, A_2, A_3, we have the following assertion. *Every irreducible representation of the rotation group is defined by a certain integer or half integer l. The transformations A_1, A_2, A_3 which correspond to infinitesimal rotations about the co-ordinate axes under this representation are described in the canonical basis $f_m (m=-l, -l+1, \cdots, l)$ by the formulas*

$$\begin{aligned} A_1 f_m &= -iH_1 f_m = \\ &= -\frac{i}{2}\sqrt{(l+m+1)(l-m)}\, f_{m+1} - \frac{i}{2}\sqrt{(l+m)(l-m+1)}\, f_{m-1}, \\ A_2 f_m &= -iH_2 f_m = \\ &= -\frac{1}{2}\sqrt{(l+m+1)(l-m)}\, f_{m+1} + \frac{1}{2}\sqrt{(l+m)(l-m+1)}\, f_{m-1}, \\ A_3 f_m &= -iH_3 f_m = -imf_m. \end{aligned} \tag{19}$$

The number l is called the *weight* of the irreducible representation corresponding to that value. Every irreducible representation is uniquely defined by its weight. We recall that the matrices T_g corresponding to an arbitrary rotation g in such a representation are defined from the matrices A_k with the aid of formula (6) of the present §.

We have found the form of the representation T_g under the hypothesis that it exists. It would be very laborious to verify directly that the matrices T_g, defined from the matrices A_k by formula (6), and the matrices A_k being defined by formulas (19), actually are a representation of the rotation group. We shall prove this in the sequel by actually constructing an irreducible representation for every l. Such representations will be constructed in §3 for every integer l and in §7 for every integer and half integer value of l. For the moment, we shall merely show that if there exists a representation defined by formula (6), where the A_k have the form (19), then this representation is irreducible. In fact, let us suppose that there exists a proper subspace R_1 invariant with respect to H_+, H_-, and H_3. Consider the transformation H_3 in this space. This transformation has an eigenvector $h = \Sigma_{m=-l}^{l} c_m f_m$, corresponding to the greatest eigenvalue of H_3 in the subspace R_1. It follows from the lemma on page 225 that the transformation H_+ carries the vector corresponding to the greatest eigenvalue of H_3 into 0. According to (18), we thus have

$$H_+h = \sum_{m=-l}^{l} c_m H_+ f_m = \sum_{m=-l}^{l} c_m a_{m+1} f_{m+1} = 0.$$

Since the vectors f_m are linearly independent, the coefficients of all f_m must be 0. For $m < l$, we have $a_{m+1} \neq 0$ and hence $c_m = 0$. This implies that $h = c_l f_l$. Therefore R_1 contains f_l. But if the subspace R_1 contains f_l, it must also contain $H_- f_l$, $H_-^2 f_l$ and so on, that is, $f_{l-1}, f_{l-2}, \cdots, f_{-l}$. Therefore R_1 coincides with R and is not a proper subspace.

We have proved that there is no proper subspace of R invariant with respect to H_+, H_-, and H_3, that is, with respect to A_1, A_2, and A_3. From this it follows that there is no proper subspace invariant under all T_g, since in the oposite case there would exist a proper subspace invariant under all

$$A_k = \left.\frac{\partial T(\xi_1, \xi_2, \xi_3)}{\partial \xi_k}\right|_{\xi_1=\xi_2=\xi_3=0}$$

This implies that the representation $g \to T_g$ is irreducible.

We note that the following fact, which will be useful in the sequel, follows from our proof. *The space R in which an irreducible representation T_g acts contains one, and except for scalar multiples, only one, vector f such that $H_+ f = 0$.*

4. Decomposition of a representation into irreducible parts. We now consider a reducible representation of the rotation group. Almost all of the considerations set forth in the preceding papagraph were not based on the irreducibility of the representation. Irreducibility was used only at the very end, in showing that the system of vectors f_m $(m = -l, -l+1, \cdots, l)$ is a basis in the space R. If we do not suppose that the representation is irreducible, then our argument shows only that this system of vectors is a basis for some invariant subspace R_0 of R. Consider the orthogonal complement R' of R_0, that is, the set of all vectors which are orthogonal to $f_l, f_{l-1}, \cdots, f_{-l}$. Since the transformations H_k are self-adjoint, it follows that R', as the orthogonal complement of an invariant subspace, is itself invariant under H_1, H_2, H_3. We can now repeat in the invariant subspace R' the same arguments that were used before, that is, we can choose the largest eigenvalue l_1 of the transformation H_3 in this subspace, then construct again the sequence of vectors $f'_m(-l_1 \le m \le l_1)$ starting with the eigenvector f'_{l_1}. We can then take the orthogonal complement of this subspace of R', and so on, so long as the original space R is not exhausted. We come, finally, to the following conclusion.

Let there be given a certain unitary representation of the rotation group. Then there exists a normal orthogonal basis in which the matrices $A_k = iH_k$ *have the following form:*

$$A_k = \left\| \begin{array}{ccccc} A_k^{(0)} & & & & \\ & A_k^{(1)} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & A_k^{(s)} \end{array} \right\|, \tag{20}$$

where the matrices $A_1^{(j)}$, $A_2^{(j)}$, $A_3^{(j)}$ *are defined from formula* (19) *of the preceding paragraph with* $l = l_j$. *Specifically,*

$$A_1^{(j)} = -\frac{i}{2} \left\| \begin{array}{cccccc} 0 & \alpha_{-l_j+1} & 0 & \dots & 0 & 0 \\ \alpha_{-l_j+1} & 0 & \alpha_{-l_j+2} & \dots & 0 & 0 \\ 0 & \alpha_{-l_j+2} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \alpha_{l_j} \\ 0 & 0 & 0 & \dots & \alpha_{l_j} & 0 \end{array} \right\|;$$

$$A_2^{(j)} = \frac{1}{2} \left\| \begin{array}{cccccc} 0 & -\alpha_{-l_j+1} & 0 & \dots & 0 & 0 \\ \alpha_{-l_j+1} & 0 & -\alpha_{-l_j+2} & \dots & 0 & 0 \\ 0 & \alpha_{-l_j+2} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -\alpha_{l_j} \\ 0 & 0 & 0 & \dots & \alpha_{l_j} & 0 \end{array} \right\|; \tag{21}$$

$$A_3^{(j)} = \left\| \begin{array}{cccccc} il_j & 0 & 0 & \dots & 0 & 0 \\ 0 & i(l_j-1) & 0 & \dots & 0 & 0 \\ 0 & 0 & i(l_j-2) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -i(l_j-1) & 0 \\ 0 & 0 & 0 & \dots & 0 & -il_j \end{array} \right\|$$

$$(\alpha_m = \sqrt{(l_j+m)(l_j-m+1)}\,).$$

We now make some observations which ordinarily permit one to find the irreducible parts into which a given representation is decomposed. Consider the transformation H_+. It is easy to see that every vector f for which $H_+ f = 0$ has the form

$$\alpha f_l + \alpha' f'_{l_1} + \cdots. \tag{22}$$

In particular, f_l, f'_{l_1}, and so on have this property. These last are characterized by the fact that they are eigenvectors for H_3: $H_3 f_l = l f_l$, $H_3 f'_{l_1} = l_1 f'_{l_1}, \cdots$. Proceeding from the above remarks, we can give the following rule for the construction of the basis in which the matrices A_1, A_2, A_3 have the form (20), or, as we shall say, for the decomposition of the representation into irreducible parts. We look for all solutions of the equation $H_+ f = 0$. The set of all these solutions is a subspace invariant under H_3. Consider the transformation H_3 in the subspace of vectors f for which $H_+ f = 0$ and find a complete orthogonal and normalized system of its eigenvectors.

For each of these vectors, say f'_{l_1}, one constructs the part of the basis for the entire space which corresponds to the particular invariant subspace, that is, the vectors $H_- f'_{l_1}$, $H_-^2 f'_{l_1}$, $\cdots$, $H^{2l_1} f'_{l_1}$.

We may draw the following conclusion from the foregoing. *If there is given a certain representation, then, in decomposing it into irreducible parts, one encounters representations with those values of l for which there exists a simultaneous solution of the equations*

$$H_+ f = 0, \qquad H_3 f = l f. \tag{23}$$

The representation with this value of l appears in the decomposition as many times as there are linearly independent solutions of the equations (23).

We mention also that the decomposition is not unique if a representation with a given value of l appears more than once in the decomposition. In fact, in constructing the f-sequences which together form a basis in R, we began with an orthogonal normalized system of vectors f such that $H_+ f = 0$, $H_3 f = l f$. But if one and the same l is encountered several times, then it follows that H_3 has multiple eigenvalues in the subspace of the vectors f for which $H_+ f = 0$. Thus the basis can be chosen in many different ways.

We now indicate a different method for decomposition into invariant subspaces. In this method, the representation is either irreducible or a multiple of an irreducible representation in each subspace (a representation is said to be a multiple of an irreducible representation if it can be decomposed into irreducible representations with one and the same weight). In contradistinction to the decomposition into irreducible representations, this decomposition is unique.

To construct this decomposition, consider the transformation

$$H^2 = H_1^2 + H_2^2 + H_3^2. \tag{24}$$

The transformation H^2 commutes with the transformations H_1, H_2, H_3, that is, we have the equalities

$$[H^2, H_1] = 0, \qquad [H^2, H_2] = 0, \qquad [H^2, H_3] = 0.$$

For example let us compute $[H^2, H_3]$:

$$[H_1^2, H_3] = H_1^2 H_3 - H_3 H_1^2 = H_1^2 H_3 - H_1 H_3 H_1 + H_1 H_3 H_1 - H_3 H_1^2 =$$
$$= H_1 [H_1, H_3] + [H_1, H_3] H_1 = -i H_1 H_2 - i H_2 H_1.$$

Analogously

$$[H_2^2,\ H_3] = H_2[H_2,\ H_3] + [H_2,\ H_3]\,H_2 = iH_2H_1 + iH_1H_2$$

and obviously

$$[H_3^2,\ H_3] = 0.$$

Combining these equalities, we have

$$[H_1^2 + H_2^2 + H_3^2,\ H_3] = [H^2,\ H_3] = 0.$$

In just the same way, it is proved that

$$[H^2,\ H_1] = 0; \quad [H^2,\ H_2] = 0.$$

In the case of an irreducible representation, where H_1, H_2, H_3 are defined by the formulas (19), a direct calculation shows that

$$H^2 = l\,(l+1)\,E,$$

where l is the number defining the representation. In carrying out this calculation, it is useful to note that

$$H_+H_- = (H_1 + iH_2)(H_1 - iH_2) = H_1^2 + H_2^2 + i\,(H_2H_1 - H_1H_2) = H_1^2 + H_2^2 + H_3.$$

Hence

$$H^2 = H_+H_- - H_3 + H_3^2.$$

From formulas (18), we find

$$\begin{aligned} H_+H_-f_m &= H_+\alpha_m f_{m-1} = \alpha_m^2 f_m, \\ H_3f_m &= mf_m, \\ H_3^2f_m &= m^2f_m. \end{aligned}$$

Since $\alpha_m^2 - m + m^2 = l(l+1)$, we infer that

$$H^2f_m = l\,(l+1)\,f_m.\,^{*}$$

In this way we see that *all vectors f of the space R in which an irreducible representation with weight l acts satisfy the equation*

$$H^2f = l\,(l+1)\,f. \tag{25}$$

It follows from this that the number of linearly independent solutions of this equation is a multiple of $2l + 1$.

With the aid of the transformation H^2, we can decompose an arbitrary representation into representations which are multiples of irreducible representations. This means that we can choose a basis in the space R consisting of disjoint sets of vectors such that, first, every set generates an invariant subspace, and, second, the representation is either irreducible or a multiple of an irreducible representation in this subspace. It is clear that in order to obtain a representation which is a multiple of an irreducible representation, it is necessary to find a complete system of linearly independent solutions of the equation (25) for each value of l. The set of all these systems of solutions gives a basis in which the representation decomposes into multiples of irreducible representations. As we have already recalled, this decomposition, unlike decompositions into irreducible parts, is unique.

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* The fact that $H^2 = aE$ (a a number) if the representation is irreducible can easily be proved from the fact that H^2 commutes with all of the H_k. The value of the constant a can be easily found by applying both sides of the equality $H^2 = aE$ to the vector f_{-l}.

5. Examples of representations. In conclusion, we consider a few examples of irreducible representations of the rotation group.

We first set $l = 0$. In this case, the representation is 1-dimensional and the matrices T_g are numbers. We evidently obtain such a representation by setting $T_g \equiv 1$ (the identity representation). The matrices A_k are all 0 in this case, a fact which is incidentally obvious from formula (19).

We next set $l = 1$. Then $2l+1 = 3$, that is, we must have a representation of the rotation group in 3-dimensional space. We obtain such a representation by mapping every rotation into its own matrix (this is the principal representation). Since every orthogonal matrix becomes unitary when considered as the matrix of a transformation in complex space, this representation is unitary. The canonical basis for the principal representation has the form

$$f_{-1} = \frac{e_x - ie_y}{\sqrt{2}}, \quad f_0 = e_z, \quad f_1 = \frac{-e_x - ie_y}{\sqrt{2}}, \quad *$$

where e_x, e_y, e_z are the unit vectors in the directions of the co-ordinate axes. It is easy to see that the transformations A_1, A_2, A_3 have the form in the canonical basis which is obtained from (19) with $l = 1$. We now find the matrix in this basis corresponding to a finite rotation through the angle ϕ around the axis oz. In the basis e_x, e_y, e_z this transformation has the form

$$\begin{aligned} e'_x &= e_x \cos\phi + e_y \sin\phi, \\ e'_y &= -e_x \sin\phi + e_y \cos\phi, \\ e'_z &= \phantom{-e_x \sin\phi + {}} e_z. \end{aligned}$$

Hence

$$\begin{aligned} f'_{-1} &= \frac{e'_x - ie'_y}{\sqrt{2}} = \frac{(e_x \cos\varphi + e_y \sin\varphi) - i(-e_x \sin\varphi + e_y \cos\varphi)}{\sqrt{2}} = \\ &= \frac{(e_x - ie_y)\, e^{i\varphi}}{\sqrt{2}} = e^{i\varphi} f_1, \\ f'_0 &= e'_z = e_z = f_0, \\ f'_1 &= \frac{-e'_x - ie'_y}{\sqrt{2}} = \frac{-e_x - ie_y}{\sqrt{2}}\, e^{-i\varphi} = e^{-i\varphi} f_1. \end{aligned}$$

Therefore the matrix of the rotation g_ϕ has the form

$$\left\| \begin{matrix} e^{i\varphi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\varphi} \end{matrix} \right\|.$$

We now find the components a_-, a_0, and a_+ of a vector a in the canonical basis. We use the equality $a_x e_x + e_y a_y + e_z a_z = a_- f_{-1} + a_0 f_0 + a_+ f_1$ and replace f_{-1}, f_0, and f_1 by their expressions in terms of e_x, e_y, e_z. This gives us

$$a_x = \frac{a_- - a_+}{\sqrt{2}}, \qquad a_y = -\frac{i}{\sqrt{2}}(a_- + a_+), \quad a_z = a_0$$

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* The elements of the canonical basis are normalized eigenvectors of the transformation H_3. From formula (7), §1, it is easy to show that the matrix H_3 has the form

$$\left\| \begin{matrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\| \text{ in the basis } e_x, e_y, e_z.$$

or

$$a_+ = \frac{-a_x + i a_y}{\sqrt{2}},$$

$$a_0 = a_z,$$

$$a_- = \frac{a_x + i a_y}{\sqrt{2}}.$$

Before considering other examples, we find the matrix T_g corresponding to a finite rotation about the axis oz, in the case of arbitrary weight l. From (19), we infer that the matrix A_3 corresponding to an infinitesimal rotation about the axis oz has the form

$$A_3 = \begin{Vmatrix} il & 0 & 0 & \dots & 0 \\ 0 & i(l-1) & 0 & \dots & 0 \\ 0 & 0 & i(l-2) & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -il \end{Vmatrix}.$$

As was shown in paragraph 1 of the present §, the rotation g through an angle ϕ around the axis oz has the corresponding matrix $T_g = e^{A_3\phi}$, that is

$$T_g = \begin{Vmatrix} e^{il\varphi} & 0 & 0 & \dots & 0 \\ 0 & e^{i(l-1)\varphi} & 0 & \dots & 0 \\ 0 & 0 & e^{i(l-2)\varphi} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & e^{-il\varphi} \end{Vmatrix}.$$

If the number l is a half integer, then we must associate the two matrices E and $-E$ with the identity rotation. In fact, as the angle ϕ of the rotation changes continuously from 0 to 2π, the matrix T_g changes continuously from E to $-E$. It follows that if l is a half integer, we do not obtain a single-valued continuous representation of the rotation group. We encountered a similar situation in paragraph 4, §1. There we associated two unitary matrices, differing in sign, with every rotation. To the product of two rotations, there corresponded thus the product of the matrices, again taken with the sigh + or –. We obtained in this fashion what is called a two-valued representation of the rotation group. Such representations will be studied in detail in §6. It will be shown there that an analogous correspondence can be established for all half integers l.

The case discussed in §1 corresponds to the value $l = 1/2$. For this case, let us find the matrices A_k which correspond to infinitesimal rotations about the co-ordinate axes. These matrices are unambiguously defined if we put the matrix

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

into correspondence with the identity rotation and define the sign of the corresponding matrices for small rotations by continuity (see paragraph 4, §1). In §1, we have described the matrices of the second order which correspond to finite rotations around the axes ox and oz (see formulas (18) and (19) of §1). Differentiating these matrices with respect to θ and ϕ, respectively, and setting the values of the parameters equal to 0, we find the matrices A_1 and A_3:

$$A_1=\frac{i}{2}\begin{Vmatrix}0 & 1\\ 1 & 0\end{Vmatrix},\quad A_3=\frac{i}{2}\begin{Vmatrix}1 & 0\\ 0 & -1\end{Vmatrix}.$$

Furthermore,

$$A_2=[A_3,\ A_1]=\frac{1}{2}\begin{Vmatrix}0 & -1\\ 1 & 0\end{Vmatrix}.$$

The matrix A_3 obtained in this way is diagonal and coincides with the matrix A_3 of formula (19) of the present section computed for the value $l = 1/2$. At the same time, the matrices A_1 and A_2 do not coincide with the corresponding matrices (19). This means that, while the basis used in complex 2-dimensional space consists of eigenvectors of the transformation H_3, still it differs from the canonical basis in norm. Multiply the first of the basis vectors by i and the second by $-i$. This gives us

$$\begin{aligned}\tilde{A}_1&=\begin{Vmatrix}i & 0\\ 0 & -i\end{Vmatrix}\frac{i}{2}\begin{Vmatrix}0 & 1\\ 1 & 0\end{Vmatrix}\begin{Vmatrix}-i & 0\\ 0 & i\end{Vmatrix}=-\frac{i}{2}\begin{Vmatrix}0 & 1\\ 1 & 0\end{Vmatrix},\\ \tilde{A}_2&=\begin{Vmatrix}i & 0\\ 0 & -i\end{Vmatrix}\frac{1}{2}\begin{Vmatrix}0 & -1\\ 1 & 0\end{Vmatrix}\begin{Vmatrix}-i & 0\\ 0 & i\end{Vmatrix}=\frac{1}{2}\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix},\\ \tilde{A}_3&=\begin{Vmatrix}i & 0\\ 0 & -i\end{Vmatrix}\frac{i}{2}\begin{Vmatrix}1 & 0\\ 0 & -1\end{Vmatrix}\begin{Vmatrix}-i & 0\\ 0 & i\end{Vmatrix}=\frac{i}{2}\begin{Vmatrix}1 & 0\\ 0 & -1\end{Vmatrix}.\end{aligned}\tag{26}$$

We see that the matrices $\tilde{A}_k$ coincide with the matrices defined by (19) for $l = 1/2$, that is, the new basis is indeed canonical. The second order matrix corresponding in the canonical basis to a rotation with Eulerian angles ϕ_1, θ, ϕ_2 has the following form:

$$\begin{Vmatrix}\cos\frac{\theta}{2}e^{i\frac{\varphi_1+\varphi_2}{2}} & -i\sin\frac{\theta}{2}e^{i\frac{\varphi_2-\varphi_1}{2}}\\ -i\sin\frac{\theta}{2}e^{-i\frac{\varphi_2-\varphi_1}{2}} & \cos\frac{\theta}{2}e^{-i\frac{\varphi_1+\varphi_2}{2}}\end{Vmatrix}.\tag{27}$$

Up to this point, we have considered irreducible representations of the rotation group. We shall now briefly discuss representations of the full orthogonal group, that is, the group consisting of all rotations with possible reflections. Since every element of the full orthogonal group is either a rotation g or the product gg_- of a rotation with reflection in the origin (see paragraph 1, §1), it follows that in order to describe the irreducible representations of the full orthogonal group, it suffices to specify the transformation which corresponds to reflection in the origin. Since g_-^2 is the identity rotation, it follows that $T_2 = E$. Consequently $T_{g_-} = \pm E$. Therefore, for every integer l, there exist two different irreducible representations of weight l of the full orthogonal group. For the first of these, $T_{g_-} = E$ and for the second, $T_{g_-} = -E$.

For example, if $l = 0$, we have two representations. In the first of them $T_{g_-} = E$, and neither rotations nor reflections change the transformed quantity. Such a quantity is called a *scalar*. In the second of the representations, we have $T_{g_-} = -E$, and the quantity does not change under rotations and changes its sign under reflections. Such a quantity is called a *pseudoscalar*. An example of a pseudoscalar is a determinant, constructed from the projection of three given vectors.

For $l = 1$ the quantities transformed under the representation are vectors of 3-dimensional space. Under reflections, ordinary vectors change their sign, that is, $T_{g_-} = -E$. If $T_{g_-} = E$, then a quantity which transforms by this rule is called a pseudovector. In textbooks on vector ahalysis, ordinary vectors are often called polar vectors, and pseudovectors (for example, the vector product of two polar vectors) are called axial vectors.

In the general case, a quantity which is transformed under an irreducible representation of the full orthogonal group is called an l-vector if $T_{g_-} = (-1)^l E$ and the corresponding pseudoquantity if $T_{g_-} = (-1)^{l+1} E$.

Appendix to §2. Proof of the differentiability of the matrix T_g. In constructing the matrices A_1, A_2, A_3, we made use of the differentiability of T_g with respect to the parameters ξ_1, ξ_2, ξ_3. We shall now prove this differentiability. To do this, we shall show that for every vector η, the function $T_g\eta$ is a differentiable function of g. Select an arbitrary element $\xi \in R$ and set first

$$\eta = \int f(g)\, T_g \xi\, dg,$$

where $f(g)$ is a certain function on G which is differentiable with respect to g (that is, with respect to the parameters ξ_1, ξ_2, ξ_3). The integration is taken in the sense of the invariant integral introduced in paragraph 3, §1. We note that $f(g)T_g\xi$ is a vector in R, and by its integral we understand the following: we write $f(g)T_g\xi$ in an arbitrary basis and then integrate each component.

We shall show that $T_{g_0}\eta$ is differentiable with respect to g_0. To do this, we compute $T_{g_0}\eta$. We have

$$T_{g_0}\eta = T_{g_0}\int f(g)\, T_g \xi\, dg = \int f(g)\, T_{g_0} T_g \xi\, dg = \int f(g)\, T_{g_0 g} \xi\, dg.$$

Since the integral is invariant, we have

$$T_{g_0}\eta = \int f(g_0^{-1} g)\, T_g \xi\, dg.$$

Consequently, in order to differentiate $T_{g_0}\eta$ it is necessary to differentiate the function $f(g_0^{-1} g)$. But we have supposed that this function is differentiable. Therefore $T_{g_0}\eta$ is differentiable.

We shall now show that $T_{g_0}\eta$ is differentiable for an arbitrary vector η. It is clear that if the function $T_{g_0}\eta$ is differentiable with respect to g_0 for $\eta = \eta_1$ and $\eta = \eta_2$, then it is differentiable for all vectors η which are linear combinations of η_1 and η_2. We have proved that $T_g\eta$ is differentiable for vectors of the form

$$\eta = \int f(g)\, T_g \xi\, dg. \tag{*}$$

We shall show that linear combinations of vectors of the form (*) comprise the entire space R. For this, we show that a vector orthogonal to all vectors of the form (*) is equal to zero. In fact, suppose that $(\eta_0, \eta) = 0$ for all $\eta = \int f(g) T_g \xi\, dg$, where $f(g)$ is an arbitrary differentiable function. That is,

$$\left(\eta_0, \int f(g)\, T_g \xi\, dg\right) = 0.$$

This may be rewritten as

$$\int f(g)(\eta_0, T_g\xi)\,dg = 0.$$

Since $f(g)$ is an arbitrary differentiable function, we clearly have

$$(\eta_0, T_g\xi) = 0 \text{ for arbitrary } g \text{ and arbitrary } \xi.$$

In particular, putting $g = e$, we have

$$(\eta_0, \xi) = 0 \text{ for arbitrary } \xi.$$

This implies that $\eta_0 = 0$, as we wished to prove. It follows that the function $T_g\eta$ is differentiable for all vectors η. Selecting elements of a basis for the vector η, we see that the elements of the matrix $T_g\eta$ must be differentiable.

§3. Spherical functions and representations of the rotation group

In §2, we classified all possible irreducible representations of the rotation group in 3-dimensional space. We turn now to one of the most frequently encountered realizations of these representations. This realization, which is very important for the purposes of analysis, constructs the operators T_g as transformations of functions. In this, we very naturally encounter systems of functions which are invariant with respect to rotations. These are the spherical functions.

The results of the present section will also show that there exist irreducible representations of weight l for all integral l. The method which we employ here to find spherical functions is fairly general. Thus, we shall use an analogous method in §7 to find another class of special functions.

1. Definition of spherical functions. Consider a function $f(x) = f(x_1, x_2, x_3)$. We write the rotation g in the form

$$x' = gx \quad \text{или} \quad x'_i = \sum g_{ik}x_k. \tag{1}$$

If we substitute for the x_k in $f(x_1, x_2, x_3)$ their values in terms of the x'_i as obtainable from (1), we obtain a new function $f_1(x'_1, x'_2, x'_3)$. We shall say that the function f goes into the function f_1 under the rotation g. The transformation which carries f into f_1 will be denoted by T_g. In this way, for every rotation g, there exists a transformation T_g on the functions $f(x)$. This transformation carries the function f into the function f_1, where f_1 is obtained from f by replacing x by its expression in terms of x'. We can express this as follows:

$$T_g f(x) = f_1(x), \text{ where } f_1(x) = f(g^{-1}x). \tag{2}$$

It is clear that the transformation T_g is linear: the sum of functions goes into the sum and multiplication by a number into multiplication by a number.

We shall show that the product of two transformations T_{g_1} and T_{g_2} corresponds to the product of the rotations g_1 and g_2. Consider the two rotations g_1 and g_2 taken consecutively:

$$x' = g_1x,$$
$$x'' = g_2x'.$$

As a result of the first rotation, $f(x)$ goes into

$$T_{g_1} f(x) = f(g_1^{-1}x),$$

and as a result of the second, this function goes into

$$T_{g_2}T_{g_1} f(x) = T_{g_2} f(g_1^{-1}x) = f((g_1^{-1}g_2^{-1}x)) = f((g_2g_1)^{-1}x)) = T_{g_2g_1} f(x).$$

This implies that

$$T_{g_2 g_1} = T_{g_2} T_{g_1}.$$

Since a sphere with center at the origin goes into itself under all rotations, it is convenient to limit ourselves to functions defined on the surface of such a sphere. We shall suppose for this reason that $x_1^2 + x_2^2 + x_3^2 = 1$, that is, that x lies on the surface of the unit sphere. It will often be convenient to suppose that the vector x is given by its spherical co-ordinates θ and ϕ, and to set $x_1 = \sin\theta\cos\phi$, $x_2 = \sin\theta\sin\phi$, $x_3 = \cos\theta$.

We limit ourselves also to functions $f(\theta, \phi)$ which have the property that the square of their absolute value is integrable over the surface of the sphere, and we define the scalar product for such functions by the formula

$$(f, g) = \int_0^{2\pi}\int_0^{\pi} f(\theta, \varphi)\overline{g(\theta, \varphi)}\sin\theta\, d\theta\, d\varphi.$$

The transformations T_g are unitary in the metric defined by this scalar product. In fact, since the element of surface of the sphere does not change under rotations, the integral of the product of two transformed functions is equal to the integral of the product of the two original functions. If we denote the co-ordinates of the vector x by θ, ϕ and the co-ordinates of the vector x' by θ', ϕ', then we accordingly have

$$(T_g f,\ T_g g) = \int_0^{2\pi}\int_0^{\pi} f(\theta', \varphi')\overline{g(\theta', \varphi')}\sin\theta\, d\theta\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\pi} f(\theta', \varphi')\overline{g(\theta', \varphi')}\sin\theta'\, d\theta'\, d\varphi' = (f, g).$$

This implies of course that the transformations T_g are unitary.

With the aid of the transformations just introduced, we can construct irreducible representations with arbitrary integer weights l. In order to do this, we construct finite-dimensional spaces consisting of functions, in which the transformations T_g are irreducible representations of the rotation group with a given weight l. The finite-dimensional space, for each l, consists of linear combinations of $2l + 1$ functions $f_m(x)$ $(-l \leq m \leq l)$.* We shall choose the functions $f_m(x)$ in such a way that they are a canonical basis for this representation. *Functions on the sphere which belong to a space in which an irreducible representation of weight l is realized are called spherical functions of the l-th order. The functions $f_m(x)$ which form a canonical basis in this space are called the principal spherical functions of the l-th order.* Since the canonical basis is defined with the help of transformations which correspond to infinitesimal rotations, we begin our construction of the functions $f_m(x)$ in paragraph 2 by finding the transformations A_1, A_2, and

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* Strictly speaking, by defining the transformations T_g and proving property (3) and that the T_g are unitary, we have defined a unitary representation of the rotation group in the space of all functions with integrable square absolute value on the surface of the sphere. It is clearly natural to pose the problem of decomposing this representation into irreducible parts. We solve the principal part of this problem by constructing the system of functions $f_m(x)$ for each l. In fact, we obtain invariant subspaces of this function space in which irreducible representations of the rotation group act. We shall solve this problem completely in paragraph 5 of this section.

A_3 which correspond under the representation T_g to infinitesimal rotations.

2. Differential operators corresponding to infinitesimal rotations. In paragraph 1, we defined the linear transformations T_g in a certain space of functions on the surface of the unit sphere in 3-dimensional space. We now construct the transformations A_1, A_2, and A_3 corresponding to infinitesimal rotations about the co-ordinate axes in this space. The functions f to which the transformations T_g. are applied will be taken here to be differentiable.*

We first find the operator A_3 which corresponds to an infinitesimal rotation about the axis oz. We must first examine a rotation g through the angle a. Since $T_g = E + aA_3 + \cdots$ (see the definition in §2), we must expand T_g in powers of a and take the coefficient of a.

We have $T_g f(x) = f(g^{-1}x)$. Hence, for the rotation g around the axis oz, we have $T_g f(\theta, \phi) = f(\theta, \phi - a)$. Expanding $f(\theta, \phi - a)$ in powers of a, we obtain

$$f(\theta, \varphi - \alpha) = f(\theta; \varphi) - \alpha \frac{\partial f(\theta, \varphi)}{\partial \varphi} + \dots \tag{4}$$

Therefore

$$A_3 f = -\frac{\partial f(\theta, \varphi)}{\partial \varphi},$$

and the operator A_3 is a differential operator, having the form

$$A_3 = -\frac{\partial}{\partial \varphi}.$$

It is easy to see that all of the operators $A_k (k=1, 2, 3)$ for the transformations T_g defined in paragraph 1 will be differential operators of the first order. In fact, for a small rotation g through an angle a around any fixed axis, $T_g f(\theta, \phi) = f(\theta', \phi')$, where θ' and ϕ' depend upon the angle a and are equal to θ and ϕ respectively for $a=0$.

Expanding $T_g f = f(\theta', \phi')$ in a power series with respect to a, we have

$$f(\theta', \varphi') = f(\theta, \varphi) + \left(\frac{\partial f}{\partial \theta}\frac{d\theta'}{d\alpha} + \frac{\partial f}{\partial \varphi}\frac{d\varphi'}{d\alpha}\right)\Big|_{\alpha=0} \alpha + \dots .$$

Consequently, *the operator A corresponding to a given infinitesimal rotation has the form*

$$\left.\begin{aligned} A &= a(\theta, \varphi)\frac{\partial}{\partial \theta} + b(\theta, \varphi)\frac{\partial}{\partial \varphi}, \\ &\textit{where} \\ a(\theta, \varphi) &= \frac{d\theta'}{d\alpha}\Big|_{\alpha=0}, \quad b(\theta, \varphi) = \frac{d\varphi'}{d\alpha}\Big|_{\alpha=0}. \end{aligned}\right\} \tag{5}$$

We now find the differential operator A_1 which corresponds to an infinitesimal rotation around the axis ox. In order to compute the functions $a(\theta, \phi)$ and $b(\theta, \phi)$ for this rotation, it is most convenient first to find the derivatives with respect to a of the Cartesian co-ordinates x_1, x_2, x_3 of the vector x. If g is a rotation through the angle a around the axis ox, then g^{-1} is a rotation through the angle $-a$ around the same axis. Hence the vector $x' = g^{-1}x$ has the co-ordinates $x_1' = x_1$, $x_2' = x_2 \cos a + x_3 \sin a$, $x_3' = -x_2 \sin a + x_3 \cos a$. The functions $\frac{dx_k}{da}\Big|_{a=0}$ therefore have the form

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* Since every unitary representation of the rotation group can be decomposed into irreducible parts and since we have established differentiability for this case (appendix to §2), it follows that an invariant subspace of functions in which an irreducible representation operates must consist of differentiable functions.

$$\left.\frac{dx_1}{d\alpha}\right|_{\alpha=0} = 0, \quad \left.\frac{dx_2}{d\alpha}\right|_{\alpha=0} = x_3, \quad \left.\frac{dx_3}{d\alpha}\right|_{\alpha=0} = -x_2. \tag{6}$$

Differentiating with respect to a the equations $x_1 = \sin\theta\cos\phi$, $x_2 = \sin\theta\sin\phi$, $x_3 = \cos\theta$, which connect the Cartesian co-ordinates with the spherical co-ordinates, and using (6), we find for $a = 0$ the following equations

$$\cos\theta\cos\varphi\frac{d\theta}{d\alpha} - \sin\theta\sin\varphi\frac{d\varphi}{d\alpha} = 0,$$

$$\cos\theta\sin\varphi\frac{d\theta}{d\alpha} + \sin\theta\cos\varphi\frac{d\varphi}{d\alpha} = \cos\theta,$$

$$-\sin\theta\frac{d\theta}{d\alpha} = -\sin\theta\sin\varphi.$$

According to these relations, we have

$$\frac{d\theta}{d\alpha} = \sin\varphi \quad \text{и} \quad \frac{d\varphi}{d\alpha} = \operatorname{ctg}\theta\cos\varphi.$$

Substituting in formula (5) the values found in this way for $\left.\frac{d\theta}{da}\right|_{a=0}$ and $\left.\frac{d\phi}{da}\right|_{a=0}$. We find that A_1 *is the differential operator defined by the formula*

$$A_1 = \sin\varphi\frac{\partial}{\partial\theta} + \operatorname{ctg}\theta\cos\varphi\frac{\partial}{\partial\varphi}\ . \tag{7}$$

The operator A_2 which corresponds to an infinitesimal rotation around the axis oy can be obtained by an analogous computation. However, if we note that replacing ϕ by $\phi - \frac{\pi}{2}$ corresponds to interchanging the axes ox and oy, we can obtain A_2 by substituting $\phi - \frac{\pi}{2}$ for ϕ in (7). This gives us

$$A_2 = -\cos\varphi\frac{\partial}{\partial\theta} + \operatorname{ctg}\theta\sin\varphi\frac{\partial}{\partial\varphi}\ . \tag{8}$$

It is convenient to make our further calculations with the transformations H_+, H_-, and H_3 introduced in §2. Using the expressions for A_1, A_2, A_3, we obtain

$$\left.\begin{aligned} H_+ &= H_1 + iH_2 = iA_1 - A_2 = e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\operatorname{ctg}\theta\frac{\partial}{\partial\varphi}\right), \\ H_- &= H_1 - iH_2 = iA_1 + A_2 = e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\operatorname{ctg}\theta\frac{\partial}{\partial\varphi}\right), \\ H_3 &= iA_3 = -i\frac{\partial}{\partial\varphi}\ . \end{aligned}\right\} \tag{9}$$

3. **The differential eqation of spherical functions.** As we have already said, functions on the sphere which lie in an invariant subspace in which an irreducible representation of weight l operates are called spherical functions of the l-th order.* Functions forming the canonical basis in this subspace (that is, the eigenvectors of the transformation H_3) are called the principal spherical functions of the l-th order. Principal spherical functions are ordinarily denoted by the symbol $Y_l^m(\theta,\phi)$, where m is the number of the basis vector, that is, m is the corresponding eigenvalue of H_3 $(-l \le m \le l)$. In this way, every spherical function of the l-th order is a linear combination of these $2l+1$ principal spherical functions $Y_l^m(\theta,\phi)$. We shall obtain an explicit expression for them in the following paragraph. We explain here how they depend upon ϕ and obtain the differential equation for the spherical functions.

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* It will be shown in paragraph 4 that there exists one and only one such invariant subspace for every weight l.

The function $Y_l^m(\theta, \phi)$ is an eigenfunction of the operator H_3, with corresponding eigenvalue m. Using the expression (9) for H_3, we obtain

$$H_3Y_l^m(\theta, \varphi) = -i\frac{\partial Y_l^m(\theta, \varphi)}{\partial\varphi} = mY_l^m(\theta, \varphi).$$

From this we obtain

$$Y_l^m(\theta, \varphi) = e^{im\varphi}F_l^m(\theta). \tag{10}$$

The dependence of Y_l^m on ϕ is thus clear. We shall explore in more detail the dependence of $Y_l^m(\theta, \phi)$ upon θ a little later.

It is evident from formula (10) that we can obtain in this fashion, for integer values of l, functions which are single-valued on the entire surface of the sphere. It follows that we obtain irreducible representations of this form only for integer weights l.* Since the $Y_l^m(\theta, \phi)$ are normalized eigenfunctions of the transformation H_3, it follows that

$$\int_0^{2\pi}\int_0^{\pi} |Y_l^m(\theta, \varphi)|^2 \sin\theta\, d\theta\, d\varphi = 1. \tag{11}$$

In view of the fact that $\int_0^{2\pi} |e^{im\varphi}|\, d\varphi = 2\pi$, it is more convenient to set

$$Y_l^m(\theta, \varphi) = \frac{1}{\sqrt{2\pi}} F_l^m(\theta)\, e^{im\varphi}. \tag{10'}$$

Then the normalization (11) can be written in the form

$$\int_0^{\pi} |F_l^m(\theta)|^2 \sin\theta\, d\theta = 1. \tag{11'}$$

We turn now to the derivation of the differential equations for spherical functions and for the functions $F_l^m(\theta)$. As was shown in §2, the vectors f in a space in which an irreducible representation with weight l acts satisfy the equation

$$H^2 f = l(l+1)f,$$

where $H^2 = H_1^2 + H_2^2 + H_3^2$.

Let us find the explicit form of this equation for the present case. To do this, we note that $H_1^2 + H_2^2 = \frac{1}{2}(H_+H_- + H_-H_+)$. Substituting H_+ and H_+ from formula (9), we find that

$$H_1^2 + H_2^2 = -\frac{\partial^2}{\partial\theta^2} - \operatorname{ctg}\theta\frac{\partial}{\partial\theta} - \operatorname{ctg}^2\theta\frac{\partial^2}{\partial\varphi^2}.$$

Adding to this the expression $H_3^2 = -\dfrac{\partial^2}{\partial\phi^2}$, we perform some simple transformations and obtain

$$-H^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

Therefore, the equation $[-H^2 + l(l+1)E]f = 0$ assumes the form

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} + l(l+1)f = 0. \tag{12}$$

This equation is called *the equation of spherical functions of the l-th order*. The number of linearly independent solutions of this equation (we are interested only in solutions which are continuous and differentiable on the entire sphere) must be a multiple of $2l+1$. It will follow from the results of paragraph 4 that the number of solutions is exactly $2l+1$.

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* Suppose that l is a half integer. Then, starting from a point (θ, ϕ) and varying ϕ continuously from ϕ to $\phi + 2\pi$, we would return to the same point but with the value of the function changed in sign.

We turn now to the principal spherical functions. Substituting the expression for $Y_l^m(\theta, \phi)$ given in (10) into the equation (12), we obtain an ordinary differential for the function $F_l^m(\theta)$. Namely,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dF_l^m}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]F_l^m(\theta)=0. \tag{13}$$

Introducing a new independent variable $\mu = \cos\theta$ and replacing $F_l^m(\theta)$ by $P_l^m(\mu)$, we obtain

$$[(1-\mu^2)P_l^{m\prime}(\mu)]'+\left[l(l+1)-\frac{m^2}{1-\mu^2}\right]P_l^m(\mu)=0. \tag{13'}$$

We finally come then to the result that *the principal spherical functions have the form*

$$Y_l^m(\theta,\varphi)=\frac{1}{\sqrt{2\pi}}e^{im\varphi}P_l^m(\cos\theta), \tag{14}$$

where $P_l^m(\mu)$ *satisfies the equation* (13′). In the following paragraph, we shall obtain an explicit expression for $P_l^m(\mu)$.

4. An explicit expression for spherical functions. In this paragraph, we shall obtain an explicit expression for the principal spherical functions. In carrying out this development, we shall simultaneously show that for every integer l, there exists only one invariant subspace of functions in which the irreducible representation of weight l is realized.

In order to find the canonical basis for the irreducible representation of weight l, we begin, just as in §2, with a simultaneous solution of the equations

$$H_3 f = lf,$$
$$H_+ f = 0,$$

that is, we begin with a determination of the function $Y_l^l(\theta, \phi)$ (the eigenfunction of the transformation H_3 which has the largest corresponding eigenvalue).

The first of these equations, as in paragraph 3, shows that $Y_l^l(\theta, \phi)$ has the form

$$Y_l^l(\theta,\varphi)=\frac{1}{\sqrt{2\pi}}e^{il\varphi}F_l^l(\theta).$$

Substituting this function in the second equation and dividing by $e^{i(l+1)\phi}$, we obtain the following differential equation for $F_l^l(\theta)$:

$$\frac{dF_l^l(\theta)}{d\theta}-l\operatorname{ctg}\theta\, F_l^l(\theta)=0.$$

The general solution of this equation has the form

$$F_l^l(\theta)=C\sin^l\theta. \tag{15}$$

It is evident from this that there exists only one function (barring constant multiples) among the eigenfunctions of the operator H_3 with eigenvalue l which satisfies the equation $H_+ f = 0$. Therefore, *for every l there exists only one irreducible representation of weight l,* since in the opposite case the equation $H_+ Y_l^l = 0$ would possess for some l at least two linearly independent solutions of the form

$e^{il\phi}\,\Phi(\theta)$.

Before determining $Y_l^m(\theta, \phi)$ for $m < l$, let us normalize the function $F_l^l(\theta) = C\sin^l\theta$. That is, let us choose the constant C so that the normalization condition (11′) is satisfied:

$$\int_0^\pi |F_l^l(\theta)|^2 \sin\theta \, d\theta = 1.$$

Computing this integral, we obtain

$$C^2 \int_0^\pi \sin^{2l+1}\theta \, d\theta = C^2 \cdot 2^{2l+1} \frac{[l!]^2}{(2l+1)!} = 1,$$

from which

$$C = \pm \frac{1}{2^l l!} \sqrt{\frac{2l+1}{2}} \sqrt{2l!}.$$

One ordinarily chooses the sign of C so that

$$C = (-1)^l \frac{1}{2^l l!} \sqrt{\frac{2l+1}{2}} \sqrt{2l!}.$$

It follows that

$$Y_l^l(\theta, \varphi) = \frac{(-1)^l}{\sqrt{2\pi} \cdot 2^l \cdot l!} \sqrt{\frac{2l+1}{2}} \sqrt{2l!}\, e^{il\varphi} \sin^l\theta = \frac{C}{\sqrt{2\pi}} e^{il\varphi} \sin^l\theta.$$

We now find the remaining functions $f_m = Y_l^m(\theta, \phi)$ of the canonical basis. We employ for this purpose the formula $H_- f_m = a_m f_{m-1}$, where $a_m = \sqrt{(l+m)(l-m+1)}$. Since

$$H_- = e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i \operatorname{ctg}\theta \frac{\partial}{\partial\varphi}\right),$$

it follows that

$$e^{-i\varphi}\left(-\frac{\partial Y_l^m}{\partial\theta} + i \operatorname{ctg}\theta \frac{\partial Y_l^m}{\partial\varphi}\right) = \alpha_m Y_l^{m-1}.$$

In this expression, we replace $Y_l^m(\theta, \phi)$ by $\frac{1}{\sqrt{2\pi}} e^{im\phi} F_l^m(\theta)$ and divide by $\frac{1}{\sqrt{2\pi}} e^{i(m-1)\phi}$. This gives us a recurrence relation for $F_l^m(\theta)$, as follows:

$$-\frac{dF_l^m(\theta)}{d\theta} - m \operatorname{ctg}\theta F_l^m(\theta) = \alpha_m F_l^{m-1}(\theta).$$

As above, setting $\cos\theta = \mu$ and writing $F_l^m(\theta)$ as $P_l^m(\mu)$, we obtain the relation

$$\sqrt{1-\mu^2}\left(\frac{dP_l^m(\mu)}{d\mu} - m\frac{\mu}{1-\mu^2} P_l^m(\mu)\right) = \alpha_m P_l^{m-1}(\mu). \tag{16}$$

Since we know $P_l^l(\mu)$, this formula gives us the possibility of computing each $P_l^m(\mu)$ in turn. For this purpose, we make the substitution

$$P_l^m(\mu) = (1-\mu^2)^{-\frac{m}{2}} u_m(\mu). \tag{17}$$

We then obtain from (16) a relation which, after division by $(1-\mu^2)^{-\frac{m-1}{2}}$, assumes the simple form

$$u_{m-1}(\mu) = \frac{1}{\alpha_m} \frac{du_m}{d\mu}. \tag{18}$$

We see from formula (15) that $P_l^l(\mu) = C(1-\mu^2)^{\frac{l}{2}}$. This implies that $u_l(\mu) = C(1-\mu^2)^l$.

It follows now from formula (18) that

$$u_{l-1}(\mu)=\frac{C}{\alpha_l}\frac{d(1-\mu^2)^l}{d\mu},\quad u_{l-2}(\mu)=\frac{C}{\alpha_l\alpha_{l-1}}\frac{d^2(1-\mu^2)^l}{d\mu^2},\ \ldots,$$

$$u_m(\mu)=\frac{C}{\alpha_l\alpha_{l-1}\,\cdots\,\alpha_{m+1}}\frac{d^{l-m}(1-\mu^2)^l}{d\mu^{l-m}}.$$

If we substitute the values just found for $u_m(\mu)$ in formula (17), we obtain the expression

$$P_l^m(\mu)=\frac{C}{\alpha_l\alpha_{l-1}\,\cdots\,\alpha_{m+1}}(1-\mu^2)^{-\frac{m}{2}}\frac{d^{l-m}(1-\mu^2)^l}{d\mu^{l-m}}.$$

Here $m=l,\ l-1,\ l-2,\ \cdots$. We note that for $m\leq -l-1$, we obtain $P_l^m(\mu)=0$, as one would expect.

We now replace C and the a_m by their values as computed and put $(-1)^l$ under the differentiation sign. We obtain finally

$$P_l^m(\mu)=\sqrt{\frac{(l+m)!}{(l-m)!}}\sqrt{\frac{2l+1}{2}}\frac{1}{2^l\cdot l!}(1-\mu^2)^{-\frac{m}{2}}\frac{d^{l-m}(\mu^2-1)^l}{d\mu^{l-m}}. \tag{19}$$

In particular, the function $P_l^0(\mu)$, which is often denoted simply by the symbol $P_l(\mu)$, has the form

$$P_l(\mu)=\sqrt{\frac{2l+1}{2}}\frac{1}{2^l\cdot l!}\frac{d^l(\mu^2-1)^l}{d\mu^l}. \tag{20}$$

The polynomial $P_l(\mu)$ is called the normalized Legendre polynomial of the l-th order, and the functions $P_l^m(\mu)$ are called the normalized associated Legendre functions.

We have therefore proved the following theorem. *The principal spherical functions of the l-th order have the form*

$$Y_l^m(\theta,\varphi)=\frac{1}{\sqrt{2\pi}}e^{im\varphi}P_l^m(\cos\theta),$$

where the functions $P_l^m(\mu)$ are defined by formula (19). Linear combinations of the functions $Y_l^m(\theta,\phi)$ with a given fixed l form a $(2l+1)$-dimensional space of functions which is invariant with respect to rotations of the sphere and in which is realized the irreducible representation of the rotation group with weight l.

In conclusion, we derive recurrence relations for polynomials and functions of Legendre with one and the same value of l. Two recurrence relations, in which the functions $P_l^m(\mu)$ and their first derivatives enter, are contained in the formulas of transformation for the principal spherical functions: $H_-Y_l^m(\theta,\phi)=a_mY_l^{m-1}(\theta,\phi)$ and $H_+Y_l^m=a_{m+1}Y_l^{m+1}$. The first of these formulas has been employed already, at the beginning of this paragraph, for obtaining a recurrence relation. To do this, we wrote

$$Y_l^m(\theta,\varphi)=\frac{e^{im\varphi}}{\sqrt{2\pi}}P_l^m(\cos\theta)$$

and obtained the formula

$$\sqrt{1-\mu^2}\frac{dP_l^m}{d\mu}=m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m(\mu)+\sqrt{(l+m)(l-m+1)}\,P_l^{m-1}(\mu). \tag{21}$$

The analogous relation $H_+Y_l^m=a_{m+1}Y^{m+1}$ gives a different recurrence formula

$$-\sqrt{1-\mu^2}\frac{dP_l^m}{d\mu}=m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m(\mu)+\sqrt{(l+m+1)(l-m)}\,P_l^{m+1}(\mu). \tag{22}$$

Combining formulas (21) and (22), we obtain a connection between three consecutive normalized Legendre functions:

$$\sqrt{(l+m+1)(l-m)}\,P_l^{m+1}(\mu)+2m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m(\mu)+$$
$$+\sqrt{(l+m)(l-m+1)}\,P_l^{m-1}(\mu)=0. \tag{23}$$

In §7, we shall establish other recurrence relations between associated functions and polynomials of Legendre with different values of l.

Formula (22) can be used to obtain a different and more expanded expression for the associated Legendre functions. Setting $m=0$ in this formula, we find that

$$P_l^1(\mu)=-\frac{1}{\alpha_1}(1-\mu^2)^{\frac{1}{2}}\frac{dP_l}{d\mu}.$$

In general, we set

$$P_l^m(\mu)=(1-\mu^2)^{\frac{m}{2}}V_m(\mu). \tag{24}$$

Then formula (22) gives a simple connection between the functions $V_m(\mu)$:

$$V_{m+1}(\mu)=-\frac{1}{\alpha_{m+1}}\frac{dV_m(\mu)}{d\mu}.$$

Since $V_0(\mu)=\sqrt{\frac{2l+1}{2}}\cdot\frac{1}{2^l\cdot l!}\frac{d^l(\mu^2-1)^l}{6d\mu^l}$, we find from this that

$$V_m(\mu)=(-1)^m\frac{1}{\alpha_1\alpha_2\ldots\alpha_m}\sqrt{\frac{2l+1}{2}}\frac{1}{2^l\cdot l!}\frac{d^{l+m}(\mu^2-1)^l}{d\mu^{l+m}}.$$

Substituting $V_m(\mu)$ in (24) and replacing α_m by their values, we obtain the following expression for the associated Legendre functions

$$P_l^m(\mu)=(-1)^m\sqrt{\frac{(l-m)!}{(l+m)!}}\sqrt{\frac{2l+1}{2}}\frac{1}{2^l\cdot l!}(1-\mu^2)^{\frac{m}{2}}\frac{d^{m+l}(\mu^2-1)^l}{d\mu^{m+l}}. \tag{25}$$

Comparing (19) and (25), we see that, except for a constant factor, one of them goes into the other when m is replaced by $-m$. It follows immediately that

$$P_l^m(\mu)=(-1)^m P_l^{-m}(\mu).$$

That is, the normalized associated Legendre functions with values of m differing in sign are proportional to each other.

Finally, we note that if we replace

$$\sqrt{\frac{2l+1}{2}}\frac{1}{2^l\cdot l!}\frac{d^l(\mu^2-1)^l}{d\mu^l}$$

in formula (25) by $P_l(\mu)$, then we obtain

$$P_l^m(\mu)=(-1)^m\sqrt{\frac{(l-m)!}{(l+m)!}}(1-\mu^2)^{\frac{m}{2}}\frac{d^mP_l(\mu)}{d\mu^m}. \tag{26}$$

This formula is the well-known expression of the normalized associated Legendre functions by means of the normalized Legendre polynomial of the same order.

5. Expansion of functions on the sphere in spherical functions. Up to this point, we have constructed separate irreducible representations. We know from §1, paragraph 5, that every unitary representation of the rotation group, even if it is infinite-dimensional, can be decomposed into irreducible representations. At the beginning of this section, we defined the infinite-dimensional space of functions on the surface of the sphere with integrable square absolute vlaue. The translation operators T_g yield a unitary representation of the rotation group in this space.

The general theorem on the decomposition of representations into irreducible parts implies in the present case that every function $f(\theta, \phi)$ on the surface of the sphere with integrable square absolute value can be expanded in a series

$$f = \phi_0 + \phi_2 + \cdots + \phi_l + \cdots,$$

where the series converges in the mean-square sense and every element of the series lies in an invariant subspace in which an irreducible representation operates.

We already made clear that each of these invariant subspaces consists of spherical functions of a fixed order, that is,

$$\varphi_l = \sum_{m=-l}^{l} C_l^m Y_l^m(\theta, \varphi).$$

It follows from the general theorems of §3 that the functions $Y_l^m(\theta, \phi)$ form an orthogonal system on the surface of the sphere. In fact, spherical functions with different values of l are orthogonal since they lie in subspaces in which inequivalent irreducible representations operate. For a single value of l, but with different values of m, these functions are orthogonal as elements of the canonical basis.

It has thus been proved that *an arbitrary function on the surface of the sphere with integrable square absolute value can be expanded in a series, which converges in the mean square sense, of orthogonal spherical functions.* Therefore the system of spherical functions on the surface of the sphere is complete.

Expansion of functions in a series of spherical functions is useful in many questions of mathematical physics, in view of the fact that this expansion is invariant under rotations. For this reason, spherical functions play the same rôle in problems connected with the surface of the sphere that trigonometric functions play in problems connected with the circle.

§4. The product of representations

In this section, we shall indicate a procedure whereby two representations of the rotation group can be combined to give a third, called their product. It turns out that many important representations are products of very simple ones. Thus, for example, the tensor representations, to which the following section is devoted, are products of irreducible representations with $l = 1$, while spinor representations (see §6) are products of irreducible representations with $l = 1/2$.

We shall also show how to decompose the product of two irreducible representations into irreducible representations.

1. Definition of the product of representations. Before defining the product of representations, we must introduce the product of spaces. Let R_1 be a p-dimensional Euclidean space with an orthogonal and normalized basis $e_1, e_2, \cdots, e_p$. Let R_2 be a q-dimensional Euclidean space with an orthogonal and normalized basis $f_1, f_2, \cdots, f_q$: Consider all of the pairs $e_i f_k$. Their linear combinations with arbitrary coefficients

$$h = \sum_{\substack{i,\, k=1}}^{\substack{k=q \\ i=p}} a_{ik} e_i f_k$$

will be taken as vectors of a new space R. The space R constructed in this way is called the *product of the spaces* R_1 *and* R_2 and is denoted by the symbol $R_1 \times R_2$. Thus a vector h of the space R is described by pq numbers

$$a_{ik} \quad (i=1, 2, \cdots, p;\ k=1, 2, \cdots q).$$

Thus R has dimension pq. Let $e = \Sigma_i \mu_i e_i$ be an arbitrary element of R_1 and let $f = \Sigma_k \lambda_k f_k$ be an arbitrary element of R_2. Then the symbol ef will be defined as the element of the space $R_1 \times R_2$ equal to $\Sigma_{i,\,k} \mu_i \lambda_k e_i f_k$.

We define the scalar product of two vectors $h' = \Sigma_{i,\,k} a'_{ik} e_i f_k$ and $h'' = \Sigma_{i,k} a''_{ik} e_i f_k$ in the space $R_1 \times R_2$ by the formula

$$(h', h'') = \sum_{i,\,k} a'_{ik} \overline{a''_{ik}}, \tag{1}$$

that is, we shall take the basis $e_i f_k$ of the space $R = R_1 \times R_2$ as being orthogonal and normalized.

One can define the product of three, four, $\cdots$ spaces in just the same way.

In the sequel, we shall frequently encounter the product of 3-dimensional space with itself. Each of its elements is given by a system of 9 numbers a_{ik} $(i, k=1, 2, 3)$.

The product of three 3-dimensional spaces is a 27-dimensional space, each element of which is given by a system of 27 numbers

$$a_{ikl} \quad (i, k, l = 1, 2, 3).$$

Analogously, the product of r 3-dimensional spaces is a 3^r-dimensional space, each element of which is described by a system of 3^r numbers

$$a_{i_1 i_2 \cdots i_r} \quad (i_1, i_2, \cdots, i_r = 1, 2, 3).$$

We turn now to the definition of a product of representations. Consider two representations of the 3-dimensional rotation group: a representation by matrices U_g acting in a p-dimensional space R_1 and a representation by matrices V_g acting in a q-dimensional space R_2. We can construct a new representation acting in the product R, from the representations defined in R_1 and R_2. In fact, the vectors e_i of the space R_1 go into vectors $U_g e_i$ under the transformation U_g, and the vectors f_k of R_2 go into vectors $V_g f_k$ under the transformation V_g. We then define the transformation T_g in the space $R_1 \times R_2$, corresponding to the rotation g, by the relation

$$T_g e_i f_k = U_g e_i V_g f_k. \tag{2}$$

Since the vectors $e_i f_k$ form a basis in the space $R_1 \times R_2$, we have defined the linear transformation T_g for all elements of R by defining it for the vectors $e_i f_k$. It is easy to describe the matrix of this transformation. In fact, suppose that the matrix U_g corresponding to the rotation g in the representation in R_1 has the form $U_g = \|u_{si}\|$, that is,

$$U_g e_i = \sum_{s=1}^{p} u_{si} e_s.$$

Suppose also that the matrix V_g acting in R_2 has the form $V_g = \|v_{rk}\|$, that is

$$V_g f_k = \sum_{r=1}^{q} v_{rk} f_r.$$

We then have, according to the definition of the representation T_g,

$$T_g e_i f_k = \sum_{s=1}^{p} \sum_{r=1}^{q} u_{si} v_{rk} e_s f_r.$$

It follows from this that an element of the space $R_1 \times R_2$ which has co-ordinates a_{ik} in the basis $e_i f_k$ goes into the element with co-ordinates $a'_{sr} = \Sigma_{i,k} u_{si} v_{rk} a_{ik}$.

It is easy to show from formula (2) that the transformations T_g form a representation of the 3-dimensional rotation group, that is, that to the product of rotations g_1 and g_2, there corresponds the product of the transformations T_{g_1} and T_{g_2}.

We summarize the above discussion as follows. *The product of representations $g \to U_g$ and $g \to V_g$, acting in the spaces R_1 and R_2 respectively, is the representation $g \to T_g$, acting in the space $R = R_1 \times R_2$. If the matrix of U_g in the e_i basis is $\|u_{si}\|$ and the matrix of V_g in the f_k basis is $\|v_{rk}\|$, then the vector h in $R_1 \times R_2$ having components a_{ik} in the $e_i f_k$ basis is transformed by T_g into a vector h', the components of which are defined by the relations*

$$a'_{sr} = \sum_{i=1}^{p} \sum_{k=1}^{q} u_{si} v_{rk} a_{ik}. \tag{3}$$

The product of an arbitrary finite number of representations can be defined in an analogous fashion.

We shall now show that the product of unitary representations U_g and V_g is also unitary. The definition of scalar product in the product of spaces (formula (1)) implies that

$$(ef, e'f') = (e, e')(f, f').$$

To prove that T_g is unitary, it suffices to show that T_g Maps the orthonormal basis $e_i f_k$ into another orthonormal basis. But this is obvious. Indeed,

$$\begin{aligned}(T_g e_i f_k, T_g e_{i'} f_{k'}) &= (U_g e_i V_g f_k, U_g e_{i'} V_g f_{k'}) = \\ &= (U_g e_i, U_g e_{i'}) (V_g f_k, V_g f_{k'}) = \delta_{ii'} \cdot \delta_{kk'},\end{aligned}$$

that is, $T_g e_i f_k$ is an orthonormal basis, and T_g is unitary.

We now consider an important example. The simplest representation of the 3-dimensional rotation group is the identical representation in 3-dimensional space, under which every rotation g is mapped into the matrix of g itself: $g \to \|g_{si}\|$. We shall compute the product of this representation with itself. The product of two 3-dimensional spaces is a 9-dimensional space, the elements of which are described by components a_{ik} $(i, k = 1, 2, 3)$. Since 3-dimensional space is transformed under g by the mateix $\|g_{is}\|$, it follows from (3) that the numbers a_{ik} are transformed under the product representation according to the equations

$$a'_{sr} = \sum_{i=1}^{3} \sum_{k=1}^{3} g_{si} g_{rk} a_{ik}.$$

The product of r such representations is defined in like manner. An element of the product of r 3-dimensional spaces is described by a system of 3^r numbers $a_{i_1 i_2 \cdots i_r}$ $(i_1, i_2, \cdots, i_r = 1, 2, 3)$. Under the product of the representation

$g \to \|g_{si}\|$ with itself r times, these numbers are transformed according to the equations

$$a'_{i'_1 i'_2 \dots i'_r} = \sum_{i_1, i_2, \dots, i_r = 1}^{3} g_{i'_1 i_1} g_{i'_2 i_2} \dots g_{i'_r i_r} a_{i_1 i_2 \dots i_r}. \tag{3'}$$

This representation is called a *tensor representation.*

The concept of a tensor is closely associated with the representation which we have just introduced. Let us suppose that a certain quantity is defined in every system of co-ordinates by means of a set of 3^r numbers $a_{i_1 \dots i_r}$. We must now show how these numbers are connected in diffetent systems of co-ordinates, For the case of a vector in 3-dimensional space, this connection is well known. Thus, if $g = \|g_{ik}\|$ is the matrix of transformation from one rectangular system of co-ordinates to another, then

$$a'_i = \sum g_{ik} a_k.$$

Suppose that there is given in every orthogonal co-ordinate system a set of numbers $a_{i_1 \dots i_r}$ $(i_1, \dots, i_r = 1, 2, 3)$. *Suppose further that under a change from one co-ordinate system to another with matrix* $g = \|g_{ik}\|$, *the set of numbers transforms according to the equations*

$$a'_{i'_1 i'_2 \dots i'_r} = \sum_{i_1, i_2, \dots, i_r = 1}^{3} g_{i'_1 i_1} g_{i'_2 i_2} \dots g_{i'_r i_r} a_{i_1 i_2 \dots i_r}.$$

Then we say that we have a tensor of rank r *in 3-dimensional Euclidean space.* In other words, a tensor of rank r is an element of the product of r 3-dimensional Euclidean spaces in which the above defined product of representations of the 3-dimensional rotation group acts. Tensors and tensor representations will be examined in more detail in the following section.

2. Transformations in a product of representations which correspond to infinitesimal rotations. We shall identify the transformations in a product of representations which correspond to infinitesimal rotations around each of the co-ordinate axes. To do this, we take the transformation T_g which corresponds to a rotation through the angle a around the k-th co-ordinate axis and find the coefficient of a in its expansion in powers of a (see §2). This coefficient A_k is the transformation corresponding to an infinitesimal rotation around the given axis.

We know that

$$T_g e_i f_k = U_g e_i V_g f_k.$$

Also, under a rotation through the angle a around a fixed axis, we have

$$U_g e_i = e_i + \alpha A e_i + \dots,$$
$$V_g f_k = f_k + \alpha A f_k + \dots,$$

where the letter A is used to denote each of the transformations corresponding to a given infinitesimal rotation about the several co-ordinate axes. Substituting these expressions in the definition of $T_g e_i f_k$, we obtain

$$T_g e_i f_k = (e_i + \alpha A e_i + \dots)\ (f_k + \alpha A f_k + \dots) = e_i f_k + \alpha\,(A e_i f_k + e_i A f_k) + \dots,$$

where the dots, as usual, indicate expressions of order higher than the first in a. Since this formula gives a resolution of $T_g e_i f_k$ in powers of a, we find that

$$A(e_i f_k) = Ae_i f_k + e_i A f_k. \tag{4}$$

Formula (4) gives the rule for finding the transformation in the product of representations which corresponds to an infinitesimal rotation about an arbitrary co-ordinate axis. There is an obvious analogy to the rule for differentiating a product. A similar rule holds for products of more than two representations.

3. **The product of two irreducible representations.** We suppose now that the representations $g \to U_g$ and $g \to V_g$, acting in the spaces R_1 and R_2 respectively, are irreducible. The product of two irreducible representations is usually reducible. We shall explain how it may be resolved into a product of irreducible representations.

Consider the irreducible representation $g \to U_g$ with weight l_1, acting in the space R_1 and the irreducible representation $g \to V_g$ with weight l_2 acting in the space R_2. In the spaces R_1 and R_2, choose canonical bases $e_{-l_1}, e_{-l_1+1}, \cdots, e_{l_1}$ and $f_{-l_2}, f_{-l_2+1}, \cdots, f_{l_2}$ (that is, bases consisting of normalized eigenvectors of the transformation $H_3 = iA_3$ in these subspaces). We then have

$$H_3 e_{m_1} = m_1 e_{m_1} \qquad (-l_1 \leqslant m_1 \leqslant l_1),$$

and

$$H_3 f_{m_2} = m_2 f_{m_2} \qquad (-l_2 \leqslant m_2 \leqslant l_2),$$

as formula (19) of §2 shows. We consider the transformation H_3 for the product of the transformations U_g and V_g. In accordance with formula (4) above, we have

$$\begin{aligned} H_3(e_{m_1} f_{m_2}) &= H_3 e_{m_1} f_{m_2} + e_{m_1} H_3 f_{m_2} = m_1 e_{m_1} f_{m_2} + m_2 e_{m_1} f_{m_2} = \\ &= (m_1 + m_2) e_{m_1} f_{m_2}. \end{aligned} \tag{5}$$

Therefore the basis $e_{m_1} f_{m_2}$ of the space $R_1 \times R_2$ is a system of orthonormal eigenvectors of the transformation H_3 with eigenvalues $m = m_1 + m_2$.

We now determine the irreducible parts into which the product of these two representations can be factored. We first determine the eigenvalues of the transformation H_3 in the space $R_1 \times R_2$ and their multiplicity.

Since m_1 varies from $-l_1$ to l_1 and m_2 varies from $-l_2$ to l_2, it follows from (5) that the eigenvalues m of the transformation H_3 in $R_1 \times R_2$ are the numbers $l_1 + l_2, l_1 + l_2 - 1, \cdots, -l_1 - l_2$.

To find the multiplicity of each of the eigenvalues m, it is necessary to find the number of vectors $e_{m_1} f_{m_2}$ for which $m_1 + m_2 = m$. Since the vector $e_{-m_1} f_{-m_2}$ belongs to the basis if $e_{m_1} f_{m_2}$ does, it follows that the multiplicity is the same for the eigenvalue $-m$ as for m itself. Hence we need only determine the multiplicity for non-negative eigenvalues m.

For the sake of definiteness, let us suppose that $l_1 \leq l_2$. It is obvious that there is only one eigenvector, namely $e_{l_1} f_{l_2}$, with eigenvalue $m = l_1 + l_2$. For the eigenvalue $m = l_1 + l_2 - 1$, there are two linearly independent eigenvectors $e_{l_1 - 1} f_{l_2}$ and $e_{l_1} f_{l_2 - 1}$. For the eigenvalue $m = l_1 + l_2 - 2$, there are 3 linearly independent eigenvectors, and so on. Finally, for the eigenvalue $m = l_2 - l_1$, there are $2l_1 + 1$ eigenvectors

$$e_{-l_1} f_{l_2},\ e_{-l_1+1} f_{l_2-1},\ \ldots,\ e_{l_1} f_{l_2-2l_1}. \tag{6}$$

Upon further diminution of the eigenvalue, the number of eigenvectors cannot increase, since all of the vectors e_{m_1} are already used in the series (6). In this way, we see that for every eigenvalue $m = l_2 - l_1$, $m = l_2 - l_1 - 1, \cdots, m = -(l_2 - l_1)$, there are exactly $2l_1 + 1$ linearly independent eigenvectors. For example, the $2l_1+1$ eigenvectors corresponding to the eigenvalue $m = -(l_2 - l_1)$ have the form

$$e_{-l_1}f_{-l_2+2l_1},\ e_{-l_1+1}\ f_{-l_2+2l_1-1},\ \ldots,\ e_{l_1}f_{-l_2}.$$

As the eigenvalue now diminishes from $-l_2 + l_1$ to $-l_2 - l_1$, the number of linearly independent eigenvectors diminishes from $2l_1 + 1$ to 1, so that there is only one eigenvector with eigenvalue $-l_2 - l_1$, namely, $e_{-l_1}f_{-l_2}$. If we have $l_2 < l_1$, then we obtain a completely analogous result, except that in place of $l_2 - l_1$ we write $l_1 - l_2$. The results obtained can be formulated in the following form.

The transformation H_3 *in the space* $R_1 \times R_2$ *has eigenvalues* $l_1 + l_2$, $l_1 + l_2 - 1, \cdots, -l_1 - l_2$. *The eigenvalue* $l_1 + l_2$ *has multiplicity* 1, *the eigenvalue* $l_1 + l_2 - 1$ *has multiplicity* 2, *and so on up to* $|l_2 - l_1|$. *In the range from* $|l_2 - l_1|$ *to* $-|l_2 - l_1|$, *all eigenvalues have multiplicity* $l_2 + l_1 - |l_2 - l_1| + 1$. *From* $-|l_2 - l_1|$ *to* $-l_2 - l_1$, *with each decrease by* 1 *in the eigenvalue, the multiplicity also decreases by* 1, *so that the smallest eigenvalue* $-l_2 - l_1$ *again has multiplicity* 1.*

What are the irreducible components of the given representation? Since the eigenvalues m of the transformation H_3 in the space $R_1 \times R_2$ satisfy the inequalities $-(l_1 + l_2) \leq m \leq (l_1 + l_2)$, it follows that the weights l of these irreducible components cannot in any case exceed the value $l_1 + l_2$.

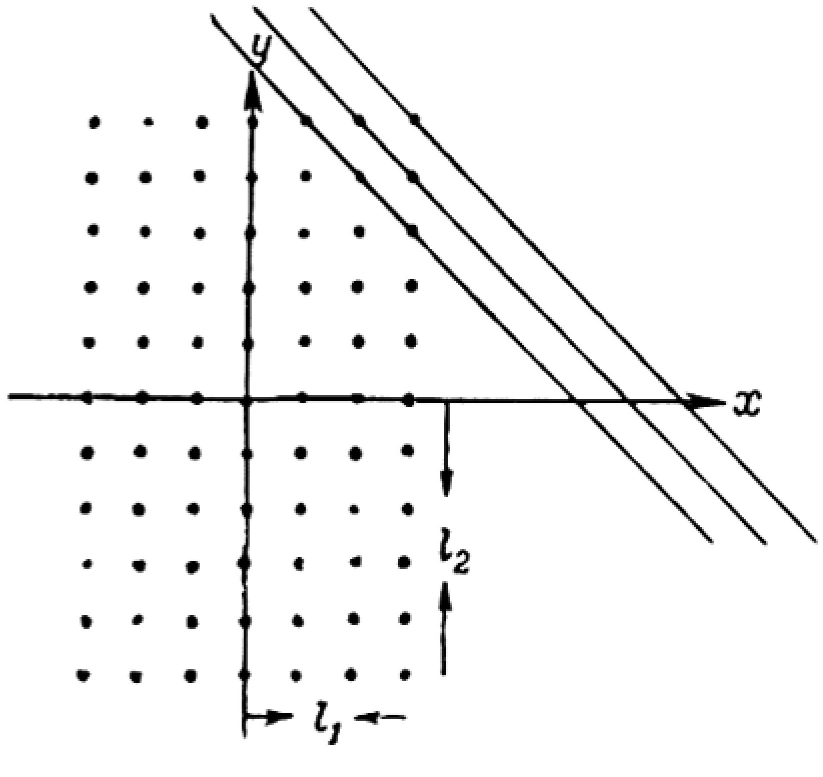

Fig. 4.

The transformation H_3 has one eigenvector $e_{l_1}f_{l_2}$ with eigenvalue $l_1 + l_2$. As we know from §2, the normalized vectors $e_{l_1}f_{l_2}$, $H_-(e_{l_1}f_{l_2})$, $H_-^2(e_{l_1}f_{l_2}), \cdots$, $H_-^{2l}(e_{l_1}f_{l_2})$ form a basis in the subspace $R^{(l)}$ in which the irreducible representation with weight $l = l_1 + l_2$ acts. Among these vectors, one can find eigenvectors of H_3 corresponding to all eigenvalues from $l_1 + l_2$ to $-(l_1 + l_2)$. Hence, if we consider the orthogonal complement R' of $R^{(l)}$, we see that in R', all eigenvalues of H_3 will have multiplicity one less than in $R = R_1 \times R_2$. In particular, the eigenvalue $l_1 + l_2$ does not occur in R', and the largest eigenvalue of H_3 in R' is $l_1 + l_2 - 1$, which has multiplicity 1.

Select an eigenvector in R' with eigenvalue $l_1 + l_2 - 1$** and construct, starting with this vector, a sequence of eigenvectors which form a basis for the subspace $R^{(2)}$, in which the irreducible representation with weight l_1+l_2-1 acts.

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* This result can also be obtained easily from Fig. 4. The marked points of the rectangle $-l_1 \leq x \leq l_1$, $-l_2 \leq y \leq l_2$ correspond to the pairs of values of m_1 and m_2 that are dealt with here. The line $x + y = m$ contains only one marked point for $m = l_1 + m_2$. As m is decreased by 1, the number of points lying on the line $x + y = m$ increases by 1, up to the value $m = |l_2 - l_1|$ (that is, for the case $l_1 < l_2$ shown in the figure, until the line reaches the upper left-hand corner of the rectangle). From this value to $m = -|l_2 - l_1|$, the line contains a constant number of marked points, and thereafter, as m decteases, the number of marked points also decreases to zero.

** One should not assume that this vector coincides with one of the basis vectors $e_{l_1-1}f_{l_2}$ or $e_{l_1}f_{l_2-1}$ corresponding to this eigenvalue. In fact, the vector $H_-(e_{l_1}f_{l_2})$, which belongs to $R^{(l)}$, is equal to $H_-e_{l_1}f_{l_2} + e_{l_1}H_-f_{l_2} = \sqrt{2l_1}e_{l_1-1}f_{l_2} + \sqrt{2l_2}e_{l_1}f_{l_2-1}$. Therefore, the vector with the same eigenvalue $l_1 + l_2 - 1$ and orthogonal to $R^{(l)}$ will also be a linear combination of the vectors $e_{l_1-1}f_{l_2}$ and $e_{l_1}f_{l_2-1}$.

Continuing this process, we construct subspaces of R, in which irreducible representations act, with weights l_1+l_2, l_1+l_2-1, l_1+l_2-2, and so on. Finally we come to a subspace in which the largest eigenvalue is equal to $|l_2-l_1|$ and has multiplicity 1. In this subspace, the transformation H_3 has eigenvalues $|l_2-l_1|$, $|l_2-l_1|-1$, $|l_2-l_1|-2$, $\cdots$, $-|l_2-l_1|$. It is easy to see that all of these eigenvalues have multiplicity 1. It follows that the representation acting in this subspace is irreducible and of weight $|l_2-l_1|$.

We have therefore obtained the following result. *The product of two irreducible representations with weights l_1 and l_2 can be decomposed into irreducible representations with weights l_1+l_2, l_1+l_2-1, $\cdots$, $|l_2-l_1|$, each of these representations being met once in the decomposition.*

4. Decomposition of the product of irreducible representations when one of them has weight 1 or $1/2$. In the preceding paragraph, we showed how to decompose the product of two irreducible representations into irreducible components, In order actually to carry out this decomposition, it is necessary to represent the canonical bases, in each of the subspaces into which $R_1\times R_2$ is decomposed, as linear combinations of the vectors $e_{m_1}f_{m_2}$. The coefficients of the corresponding linear combinations are very complicated, in the general case *. We shall compute these coefficients for the simplest and most frequently encountered cases, namely, when one of the irreducible representations, the product of which is to be decomposed, has weight $l=1$ or $l=1/2$.

Suppose that the irreducible representation $g\rightarrow U_g$ has weight $l=1$. Let $e_{-1}=e_-$, e_0, and $e_1=e_+$ be a canonical basis in the 3-dimensional space R_1.** Let the representation $g\rightarrow V_g$ have weight $l\geq 1$. Denote the vectors of the canonical basis of this representation by the symbols $f_m\ (-l\leq m\leq l)$.

Consider next the product of these two representations. A basis in the space $R_1\times R_2$, in which this representation acts, is formed by the $3(2l+1)$ vectors e_+f_m, e_0f_m, $e_-f_m\ (-l\leq m\leq l)$. By the results of paragraph 3, we know that the space $R_1\times R_2$ can be decomposed into the sum of 3 invariant subspaces, in which irreducible representations of weights $l+1$, l, and $l-1$ act. We denote canonical bases in these subspaces, consisting of eigenvectors of the transformation H_3, by the symbols g_m^{l+1}, g_m^l, and g_m^{l-1}. The superscript is the weight of the irreducible representation acting in the given subspace, and the subscript m, as always, is the eigenvalue of H_3 to which the given eigenvector corresponds.

For fixed m, each of the vectors g_m^{l+1}, g_m^l, and g_m^{l-1} is an eigenvector of H_3 in the space $R_1\times R_2$ with corresponding eigenvalue m. Hence the vectors g_m^{l+1}, g_m^l, g_m^{l-1} are linear combinations of the basis vectors $e_{m_1}f_{m_2}$ which are eigenvectors of H_3 with the same eigenvalue m, that is e_-f_{m+1}, e_0f_m, e_+f_{m-1}. In other words, g_m^{l+1}, g_m^l, g_m^{l-1} and e_-f_{m+1}, e_0f_m, e_+f_{m-1} are different bases in a 3-dimensional subspace* of the space $R_1 \times R_2$. This subspace consists of all eigenvectors of H_3 with corresponding eigenvalue m. For every m, we must find the matrix of the transformation carrying one of these bases into the other.

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* For example, see Landau and Lifšic, *Quantum Mechanics*, Gostehizdat, 1948, or E. Condon and G. Shortley, *The theory of atomic spectra*, Cambridge University Press, 1935.

** For the connection between this basis and the ordinary basis in 3-dimensional space, see paragraph 5, §2.

We set

$$\left.\begin{aligned} e_- f_{m+1} &= c_{11}^m g_m^{l+1} + c_{12}^m g_m^l + c_{13}^m g_m^{l-1}, \\ e_0 f_m &= c_{21}^m g_m^{l+1} + c_{22}^m g_m^l + c_{23}^m g_m^{l-1}. \\ e_+ f_{m-1} &= c_{31}^m g_m^{l+1} + c_{32}^m g_m^l + c_{33}^m g_m^{l-1}. \end{aligned}\right\} \tag{7}$$

Since the vectors $e_- f_{m+1}$, $e_0 f_m$, and $e_+ f_{m-1}$ are orthonormal, as are the vectors g_m^{l+1}, g_m^l, g_m^{l-1}, (they belong to distinct orthogonal subspaces of $R_1 \times R_2$), it follows that the matrix $C^{(m)} = \|c_{ik}^m\|$ is a unitary matrix of the 3rd order.

For $m = -l - 1$, the first of the equalities (7) must be deleted, and it becomes instead the equality

$$e_- f_{-l} = c_{11}^{-l-1} g_{-l-1}^{l+1}. \tag{7'}$$

For $m = -l$, we have only two equalities:

$$\left.\begin{aligned} e_- f_{-l+1} &= c_{11}^{-l} g_{-l}^{l+1} + c_{12}^{-l} g_{-l}^l, \\ e_0 f_{-l} &= c_{21}^{-l} g_{-l}^{l+1} + c_{22}^{-l} g_{-l}^l. \end{aligned}\right\} \tag{7''}$$

Analogous changes in (7) must be made for $m = l + 1$ and $m = l$.

To compute the matrix $C^{(m)}$, we apply the transformations H_+ and H_- to both sides of the equalities (7). We then obtain recurrence relations from which $C^{(m)}$ can be computed.

We find first the first line of the matrix $C^{(m)}$. For this purpose, we apply H_- to the first of the equalities (7). Recalling from §2 that $H_- f_m = a_m^l f_{m-1}$, where $a_m^l = \sqrt{(l+m)(l-m+1)}$, we obtain from the left-hand side of the first equality in (7) the expression

$$H_-(e_- f_{m+1}) = e_- H_- f_{m+1} = a_{m+1}^l e_- f_m = a_{m+1}^l (c_{11}^{m-1} g_{m-1}^{l+1} + c_{12}^{m-1} g_{m-1}^l + c_{13}^{m-1} g_{m-1}^{l-1}).$$

On the other hand, applying H_- to the right-hand side of the same equality, we obtain

$$c_{11}^m a_m^{l+1} g_{m-1}^{l+1} + c_{12}^m a_m^l g_{m-1}^l + c_{13}^m a_m^{l-1} g_{m-1}^{l-1}.$$

These expressions must be equal. Equating coefficients of the various vectors g and replacing the numbers a_m^l by their values, we obtain the recurrence relations

$$\left.\begin{aligned} c_{11}^m \sqrt{l-m+2} &= c_{11}^{m-1} \sqrt{l-m}, \\ c_{12}^m \sqrt{(l+m)(l-m+1)} &= c_{12}^{m-1} \sqrt{(l+m+1)(l-m)}, \\ c_{13}^m \sqrt{l+m-1} &= c_{13}^{m-1} \sqrt{l+m+1}. \end{aligned}\right\} \tag{8}$$

Since the vectors $e_- f_{-l}$ and g_{-l-1}^{l+1} are normalized, it follows from formula (7′) that $|c_{11}^{-l-1}| = 1$. Setting $c_{11}^{-l-1} = 1$,** we can apply the first of the formulas (8) to find that

$$c_{11}^m = \sqrt{\frac{l-m}{l-m+2}}\, c_{11}^{m-1} = \sqrt{\frac{(l-m)(l-m+1)}{(l-m+2)(l-m+3)}}\, c_{11}^{m-2} = \ldots =$$
$$= \sqrt{\frac{(l-m)(l-m+1)}{(l-n)(l-n+1)}}\, c_{11}^n = \ldots = \sqrt{\frac{(l-m)(l-m+1)}{(2l+1)(2l+2)}}\, c_{11}^{-l-1},$$

---

* Note however that if $m = l + 1$, then this space is 1-dimensional, and if $m = l$ or $m = -l$, then it is 2-dimensional.

** Since each of the three vectors g_m^{l+1}, g_m^l, g_m^{l-1} is determined up to a scalar multiple of absolute value 1, we can fix this multiple by choosing in a determined way one coefficient in each column of the matrix $C^{(m)}$ (for some one value of m).

i.e., that

$$c^m_{11} = \sqrt{\frac{(l-m)(l-m+1)}{(2l+1)(2l+2)}}\,.$$

Setting $m = -l$ and using the fact that the matrix of the transformation (7″) is unitary, i.e., that $|c^{-l}_{11}|^2 + |c^{-l}_{12}|^2 = 1$, we find further that $|c^{-l}_{12}| = \sqrt{\frac{1}{l+1}}$. Again, setting $c^{-l}_{12} = \sqrt{\frac{1}{l+1}}$ (see the last footnote), we find, with the aid of the second of the relations (8) that

$$c^m_{12} = \sqrt{\frac{(l+m+1)(l-m)}{2l(l+1)}}\,.$$

Following the same path, we use the third of the formulas (8) to show that

$$c^m_{13} = \sqrt{\frac{(l+m)(l+m+1)}{2l(l+1)}}\,.$$

(Here we use the fact that $|c^{-l+1}_{13}|^2 = 1 - |c^{-l+1}_{12}|^2 - |c^{-l+1}_{11}|^2 = \frac{1}{l(2l+1)}$ and, as above, we have set $c^{-l+1}_{13} = \sqrt{\frac{1}{l(2l+1)}}$. In this way, we have defined all of the elements of the first line of the matrix $C^{(m)}$).

To obtain the second line of this matrix, we apply the transformation H_+ to the first of the equalities (7). We then obtain

$$\sqrt{2}e_0f_{m+1} + \alpha^l_{m+2}e_-f_{m+2} = c^m_{11}\alpha^{l+1}_{m+1}g^{l+1}_{m+1} + c^m_{12}\alpha^l_{m+1}g^l_{m+1} + c^m_{13}\alpha^{l-1}_{m+1}g^{l-1}_{m-1}\,.$$

Replacing each of the vectors of the left-hand side by their expressions given in (7), we obtain simple relations which express the elements of the second row of the matrix $C^{(m)}$ in terms of the already determined elements of the first row. In this way, c^m_{21}, c^m_{22}, and c^m_{23} are defined uniquely. Furthermore, applying the transformation H_+ to the second of the equalities (7), we express, in much the same way, the elements of the third row of the matrix $C^{(m)}$ by means of elements of the second, and therefore we can find c^m_{31}, c^m_{32}, c^m_{33}. Finally, we obtain the following result

$$C^{(m)} = \begin{Vmatrix} \sqrt{\frac{(l-m)(l-m+1)}{(2l+1)(2l+2)}} & \sqrt{\frac{(l+m+1)(l-m)}{2l(l+1)}} & \sqrt{\frac{(l+m)(l+m+1)}{2l(l+1)}} \\ \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(l+1)}} & \frac{m}{\sqrt{l(l+1)}} & -\sqrt{\frac{(l+m)(l-m)}{l(2l+1)}} \\ \sqrt{\frac{(l+m)(l+m+1)}{(2l+1)(2l+2)}} & -\sqrt{\frac{(l+m)(l-m+1)}{2l(l+1)}} & \sqrt{\frac{(l-m)(l-m+1)}{2l(2l+1)}} \end{Vmatrix}. \tag{9}$$

Since this matrix is orthogonal, its inverse is its transpose, and hence the vectors g^{l+1}_m, g^l_m, g^{l-1}_m are expressed in terms of e_-f_{m+1}, e_0f_m, e_+f_{m-1} by use of the columns of this same matrix.

We next consider the product of an irreducible representation with $l = 1/2$ and an arbitrary irreducible representation with weight l. This representation can be decomposed into a representation with weight $l + 1/2$ (we denote the basis in the corresponding subspace by $g^{l+1/2}_m$ $(-l - 1/2 \le m \le l + 1/2)$) and a representation with weight $l - 1/2$ (basis denoted by $g^{l-1/2}_m$ $(-l + 1/2 \le m \le l - 1/2)$). As in the case treated above, we have

$$e_{\frac{1}{2}} f_{m-\frac{1}{2}} = c_{11}^{m} g_{m}^{l+\frac{1}{2}} + c_{12}^{m} g_{m}^{l-\frac{1}{2}},$$

$$e_{-\frac{1}{2}} f_{m+\frac{1}{2}} = c_{21}^{m} g_{m}^{l+\frac{1}{2}} + c_{22}^{m} g_{m}^{l-\frac{1}{2}}.$$

Applying H_+ to the first equality and H_- to the second, and setting $c_{11}^{l+½} = c_{21}^{-l-½} = 1$, we find that the matrix $C^{(m)}$ for the remaining values of m has the form

$$C^{(m)} = \left\| \begin{matrix} \sqrt{\dfrac{l+m+\frac{1}{2}}{2l+1}} & -\sqrt{\dfrac{l-m+\frac{1}{2}}{2l+1}} \\ \sqrt{\dfrac{l-m+\frac{1}{2}}{2l+1}} & \sqrt{\dfrac{l+m+\frac{1}{2}}{2l+1}} \end{matrix} \right\|. \tag{10}$$

§5. Tensors and tensor representations

In the preceding section, we defined the concept of tensor in connection with the definition of the product of representations. We said that *a tensor of rank r in 3-dimensional Euclidean space is defined by specifying in every orthogonal system of co-ordinates a set of 3^r numbers $a_{i_1 i_2 \dots i_r}$, having the property that upon going from one system of co-ordinates to another, these numbers are transformed by the formulas*

$$a'_{i'_1 i'_2 \dots i'_r} = \sum_{i_1=1}^{3} \sum_{i_2=1}^{3} \cdots \sum_{i_r=1}^{3} g_{i'_1 i_1} g_{i'_2 i_2} \cdots g_{i'_r i_r} a_{i_1 i_2 \dots i_r}, \tag{1}$$

where $\|g_{ik}\|$ is the matrix of transformation from one co-ordinate system to the other. By the matrix of the transformation we mean the matrix which expresses the new basis vectors e'_1, e'_2, e'_3 in terms of the old basis e_1, e_2, e_3 by means of the relations

$$e'_i = \sum_{k=1}^{3} g_{ik} e_k.$$

We consider the set of all tensors of rank r in a fixed system of co-ordinates, that is, all systems of 3^r numbers $a_{i_1 i_2 \dots i_r}$. They form a 3^r-dimensional space, in which addition and multiplication by a number are defined in the ordinary way. Formula (1) defines a linear transformation in this 3^r-dimensional space, carrying $a_{i_1 i_2 \dots i_r}$ into $a'_{i'_1 i'_2 \dots i'_r}$. *Upon mapping each rotation $g = \|g_{ik}\|$ into the linear transformation* (1) *we obtain a representation of the rotation group, which is identical with the representation discussed in* §4 *(the tensor representation).* As we have seen, this representation is the product of 3-dimensional representations of the rotation group.*

We note the behavior of tensors under a reflection in the origin. Under such a reflection, vectors (that is, tensors of rank 1) change their sign. It follows that tensors of rank *2* are not altered under reflection in the origin; components of a tensor of rank *3* change their sign; in general, the components of a tensor of rank r are multiplied by $(-1)^r$.

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* We remark that formula (1) can be understood in two senses. First, it defines a transformation of a tensor under the change from one co-ordinate system to another. Second, it defines a mapping of one tensor onto another in the same co-ordinate system. Both points of view are quite legitimate. This is distinctly obvious in the case of tensors of rank 1, that is, vectors a_i. In fact, if $\|g_{ik}\|$ is taken as the matrix of a rotation of the co-ordinate system, then the formula $a'_{i'} = \sum g_{i'i} a_i$ is the formula of transformation of the components of a given vector. However, if $\|g_{ik}\|$ is taken to be the matrix of a rotation of space in a fixed co-ordinate system, then this formula gives the components of the rotated vector.

For $r > 1$, tensor representation are reducible. In paragraph 2 of this section, we shall decompose these representations into multiples of irreducible representations. Before doing this, we take up in paragraph 1 certain operations on tensors and their connection with the decomposition of tensor space into invariant subspaces, that is, with the decomposition of tensor representations.

1. The principal algebraic operations on tensors. Invariant subspaces. We first define the operation of contraction of a tensor. Consider a tensor of rank r, $a_{i_1 i_2 \dots i_r}$. We take those components ot this tensor for which the first two indices have the same value, and form their sum

$$\sum_{i=1}^{3} a_{iii_3i_4 \dots i_r} = b_{i_3i_4 \dots i_r}. \tag{2}$$

The expression (2) *is called the trace of the tensor with respect to the first two indices, and the operation of forming the trace is called contraction.** The operation of contraction with respect to any pair of indices is defined analogously.

It follows from formula (1) and the orthogonality of the matrix $\|g_{ik}\|$ that the trace of a tensor of rank r with respect to any pair of indices is itself a tensor of rank $r-2$. In fact,

$$\begin{aligned} b'_{i'_3i'_4 \dots i'_r} = a_{i'i'i'_3 \dots i'_r} &= g_{i'i_1}g_{i'i_2}g_{i'_3i_3} \cdots g_{i'_ri_r}a_{i_1i_2 \dots i_r} = \\ &= \delta_{i_1i_2}g_{i'_3i_3} \cdots g_{i'_ri_r}a_{i_1i_2 \dots i_r} = g_{i'_3i_3}g_{i'_4i_4} \cdots g_{i'_ri_r}a_{iii_3i_4 \dots i_r} = \\ &= g_{i'_3i_3}g_{i'_4i_4} \cdots g_{i'_ri_r}b_{i_3i_4 \dots i_r}. \end{aligned}$$

In this way, we see that

$$b'_{i'_3i'_4 \dots i'_r} = g_{i'_3i_3}g_{i'_4i_4} \cdots g_{i'_ri_r}b_{i_3i_4 \dots i_r}. \tag{3}$$

We have thus proved that the trace of a tensor is transformed in accordance with the tensor law and is therefore a tensor of rank $r-2$. In particular, the trace of a tensor of the second rank is a tensor of rank *0*, that is, it is a scalar. (A scalar is a number which does not depend upon the choice of the co-ordinate system.) A tensor of rank *3* has three traces, a_{iik}, a_{iki}, and a_{kii}, each of which is a vector.

The operation of forming the trace gives us a method of constructing subspaces of R which are invariant under the tensor representation. *In fact, the elements*

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* In the sequel, we shall follow the usual convention in computations involving tensors of omitting the summation sign in expressions where the summation is carried out with respect to an index which appears twice. With this convention, formula (1), for example, is written in the form

$$a'_{i'_1i'_2 \dots i'_r} = g_{i'_1i_1}g_{i'_2i_2} \cdots g_{i'_ri_r}a_{i_1i_2 \dots i_r},$$

and the definition of the trace in the form

$$a_{iii_3 \dots i_r} = b_{i_3 \dots i_r}.$$

The equality

$$g_{i'i_1}g_{i'i_2} = \delta_{i_1i_2}$$

in this notation is the condition that the matrix $\|g_{ik}\|$ be orthogonal.

$a_{i_1 i_2 \cdots i_r}$ *of* R *whose trace with respect to a certain pair of indices is zero form a subspace of* R *which is invariant with respect to the tensor representation.* Consider the elements $a_{i_1 i_2 \cdots i_r}$ of R for which the trace with respect to the first two indices, for example, is zero: $a_{iii_3 i_4 \cdots i_r} = b_{i_3 i_4 \cdots i_r} = 0$. It then follows from formula (3) that $a'_{i'i'i'_3 \cdots i'_r} = 0$, that is, $a'_{i'_1 i'_2 \cdots i'_r}$ belongs to the same subspace. Thus this subspace is invariant.

We next define the operation of multiplication of tensors. To do this, consider two tensors $a_{i_1 i_2 \cdots i_r}$ and $b_{j_1 j_2 \cdots j_p}$, where the ranks r and p may be either equal or different. *The product of these two tensors is the tensor* $c_{i_1 i_2 \cdots i_r j_1 j_2 \cdots j_p}$ *of rank* $r+p$, *whose components are the products of the different components of the tensors* $a_{i_1 i_2 \cdots i_r}$ *and* $b_{j_1 j_2 \cdots j_p}$:

$$c_{i_1 i_2 \cdots i_r j_1 j_2 \cdots j_p} = a_{i_1 \cdots i_r} b_{j_1 \cdots j_p}.$$

It is not hard to show that these numbers are transformed, under a change of co-ordinates, just as the components of a tensor of rank $r+p$ must be transformed. That is, the product of tensors is actually a tensor. It is also clear from formula (1) that under transformations of the tensor representation, the product of tensors $a_{i_1 i_2 \cdots i_r}$ and $b_{j_1 j_2 \cdots j_p}$ goes into the product of $a'_{i'_1 i'_2 \cdots i'_r}$ and $b'_{j'_1 j'_2 \cdots j'_p}$.

In order to construct invariant subspaces of a different kind than those described above, we introduce what is called the unit tensor of rank 2, δ_{ik}, the components of which in every system of co-ordinates are defined by the equalities

$$\delta_{ik} = \begin{cases} 0, & i \neq k, \\ 1, & i = k. \end{cases}$$

We verify that δ_{ik} is a tensor. In fact,

$$\delta'_{i'k'} = g_{i'i} g_{k'k} \delta_{ik} = g_{i'k} g_{k'k} = \begin{cases} 0, & i' \neq k', \\ 1, & i' = k'. \end{cases}$$

That is, the components of this tensor do not change under passage to a new system of co-ordinates.

We now consider tensors of rank r which are products of the unit tensor and a tensor of rank $r-2$, that is, tensors of the form

$$a_{i_1 i_2 i_3 \cdots i_r} = \delta_{i_1 i_2} b_{i_3 \cdots i_r}. \tag{4}$$

Since the transformations of tensor space which produce the tensor representation all carry the unit tensor into itself, and products into products, it follows that tensors of the form (4) also form an invariant subspace in the space R of tensor of rank r.

We shall now show that every tensor of rank r can be represented as the sum of two tensors, the first of which has the form $\delta_{i_1 i_2} b_{i_3 \cdots i_r}$ and the second of which has the property that $a_{iii_3 \cdots i_r} = 0$. In fact, we take an arbitrary tensor $a_{i_1 i_2 \cdots i_r}$ and subtract from it the tensor $\frac{1}{3}\delta_{i_1 i_2} b_{i_3 \cdots i_r}$, where $b_{i_3 \cdots i_r}$ is the trace of $a_{i_1 i_2 i_3 \cdots i_r}$ with respect to the first two indices. We obtain a tensor $c_{i_1 i_2 i_3 \cdots i_r}$ whose trace with respect to the first two indices is zero (since $\delta_{ii} = 3$). This proves our assertion. Therefore the space R of tensors of rank r can be decomposed into the sum of invariant subspaces R' and R'', where R' is the subspace of all tensors whose trace with respect to the first two indices is zero, and R'' is the space of tensors of the form (4).

An analogous decomposition, naturally, can be produced by using contraction with respect to an arbitrary pair of indices.

We next exhibit yet another method of constructing invariant subspaces of R. Let s denote an arbitrary permutation of the numbers $i_1 i_2 \cdots i_r$, carrying these numbers into $j_1 j_2 \cdots j_r$. We map the tensor $a_{i_1 i_2 \ldots i_r}$ of rank r into the tensor in which the indices are permuted, that is, $a_{j_1 j_2 \ldots j_r}$. This transformation is denoted by S. That is, we set

$$Sa_{i_1 i_2 \ldots i_r} = a_{j_1 j_2 \ldots j_r}.$$

It is evident that S is a linear transformation in the space of tensors of rank r. A transformation of this kind is called a permutation transformation. Since it makes no difference whether we first permute indices and then go over to new coordinates or carry out these operations in the opposite order, it follows that the permutation transformations commute with the transformations of the tensor representation T_g.

Consider a set of permutation transformations $S_1, S_2, \cdots, S_p$, and form the linear combination $\lambda_1 S_1 + \lambda_2 S_2 + \cdots + \lambda_p S_p$. Tensors which satisfy the equation

$$(\lambda_1 S_1 + \lambda_2 S_2 + \cdots + \lambda_p S_p) a_{i_1 i_2 \ldots i_r} = 0, \tag{5}$$

form a linear subspace of R. Since permutation transformations commute with the transformations of the representation, it follows that this subspace is invariant with respect to the tensor representation.*

We now describe a method of decomposition of the tensor representation with the aid of eigenvalues of a permutation transformation.** Since some power of an arbitrary permutation is equal to the identity, it follows that the transformation S of tensor space, if repeated a sufficient number of times, yields the identity transformation E. But this means that the eigenvalues of the transformation S are equal to certain roots of the eigenvalues of E, that is, they are roots of unity.

Let s be a certain permutation and let $s^n = s_0$, where s_0 is the identity permutation. Let the number $\epsilon_0 = 1$, ϵ_1, $\epsilon_2 = \epsilon_1^2$, $\cdots$, ϵ_{n-1}, be the n-th roots of unity. Consider, in tensor space, the subspace consisting of all tensors which satisfy the equation

$$Sa_{i_1 i_2 \ldots i_r} = \epsilon_k a_{i_1 i_2 \ldots i_r}.$$

It is known from the general theory of linear transformations that these subspaces ($k = 0, 1, \cdots, n-1$) form a complete decomposition of tensor space. In accordance with the observations made above, we see that all of these subspaces are invariant under the tensor representation. This gives us, therefore, a certain decomposition

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[*] In fact, suppose that $a_{i_1 i_2 \ldots i_r}$ belongs to the subspace defined by the equation (5), that is, $\sum_j \lambda_j S_j a_{i_1 i_2 \ldots i_r} = 0$. Then

$$\sum_j \lambda_j S_j T_g a_{i_1 \ldots i_r} = T_g \left(\sum_j \lambda_j S_j a_{i_1 i_2 \ldots i_r} \right) = 0.$$

Therefore $T_g a_{i_1 i_2 \ldots i_r}$ belongs to this same subspace.

[**] Naturally this decomposition is in general not a decomposition into irreducible representations or multiples of irreducible representations.

of tensor space into invariant subspaces.

For example, if s_1 is the transposition of the first two indices, carrying $i_1 i_2 \cdots i_r$ into $i_2 i_1 \cdots i_r$, then $S_2^2 = E$ and the eigenvalues of S_1 are ±1.

Consequently every tensor $a_{i_1 i_2 \dots i_r}$ can be represented as the sum of two tensors $b_{i_1 i_2 \dots i_r}$ and $c_{i_1 i_2 \dots i_r}$,

$$a_{i_1 i_2 \dots i_r} = b_{i_1 i_2 \dots i_r} + c_{i_1 i_2 \dots i_r},$$

where $b_{i_2 i_1 \dots i_r} = -b_{i_1 i_2 \dots i_r}$ and $c_{i_1 i_2 \dots i_r} = c_{i_2 i_1 \dots i_r}$. To show this, it suffices to set

$$b_{i_1 i_2 \dots i_r} = \frac{1}{2} (a_{i_1 i_2 \dots i_r} - a_{i_2 i_1 \dots i_r}),$$

and

$$c_{i_1 i_2 \dots i_r} = \frac{1}{2} (a_{i_1 i_2 \dots i_r} + a_{i_2 i_1 \dots i_r}).$$

We also note that a set of equations of type (5) also defines an invariant subspace of tensor space, consisting of all tensors satisfying all of these equations.

Tensors which do not change under any permutation of the indices $i_1 i_2 \cdots i_r$ *are called symmetric tensors of rank r. Tensors which remain unaltered under even permutations and which change their sign under odd permutations are called skew-symmetric.* It is clear from the foregoing that the spaces of symmetric and of skew-symmetric tensors form invariant subspaces in the space R.

We now examine in more detail the space of skew-symmetric tensors in 3-dimensional space. Under transposition of any two indices, a skew-symmetric tensor must change its sign. It is thus clear that if a given component of a skew-symmetric tensor is different from zero, then all of its indices must be distinct. Since the indices $i_1, i_2, \cdots, i_r$ can assume only the values 1, 2, and 3, it follows that skew-symmetric tensors of rank higher than 3 are equal to zero.

Let a_{ijk} be a skew-symmetric tensor of rank 3. The component a_{123} of such a tensor can be arbitrary, and then the other non-zero components are defined from a_{123} by the condition of skew-symmetry: $a_{231} = a_{312} = a_{123}$; $a_{213} = a_{132} = a_{321} = -a_{123}$. Let ϵ_{ijk} denote the skew-symmetric tensor of rank 3 such that $a_{123} = 1$. The other skew-symmetric tensors of rank 3 coincide with ϵ_{ijk} up to a scalar multiple.

It follows from formula (1) that under transformation of co-ordinates, every component of the tensor ϵ_{ijk} is multiplied by the determinant of the transformation matrix $\|g_{ik}\|$. Hence the components of ϵ_{ijk} do not change under a rotation, while under an improper orthogonal transformation, they change sign. Therefore a skew-symmetric tensor of rank 3 is a pseudo-scalar with regard to the tensor representation of the rotation group.

We now consider a skew-symmetric tensor a_{ij} of rank 2. It is clear that this tensor has 3 independent components a_{12}, a_{23}, and a_{13}. Thus such tensors form a 3-dimensional space. Let us multiply a_{ij} by $\epsilon_{i'j'k'}$ and then contract with respect to the indices ii' and jj'. We obtain a tensor of rank 1,

$$x_k = \epsilon_{ijk} a_{ij},$$

that is, a vector. It is clear that $\epsilon_{ijk} a_{ij} = 2a_{ij}$ or $\epsilon_{ijk} a_{ij} = 2a_{ij}$, according to the choice of the co-ordinate system. It follows that the components of a skew-symmetric tensor of rank 2 coincide with the components of the vector x_k up to a scalar multiple, whose sign depends upon the co-ordinate system. In passing from one system to another, the numbers a_{23}, a_{31}, a_{12} are transformed just like the components of the vector x_1, x_2, x_3, if the change of co-ordinates is a proper rotation. In the case of an improper orthogonal transformation, they also change their sign. This means in other words that *skew-symmetric tensors of rank 2 behave like pseudo-vectors with respect to the tensor representation of the rotation group.*

As an illustration, we discuss the decomposition of the tensor representation into irreducible parts, for the case of tensors of rank 2. Every tensor of rank 2 can be represented as the sum of a symmetric and a skew-symmetric tensor:

$$a_{ij} = b_{ij} + c_{ij}.$$

In fact, we may set

$$b_{ij} = \frac{1}{2}(a_{ij} - a_{ji}),$$

and

$$c_{ij} = \frac{1}{2}(a_{ij} + a_{ji}).$$

In this way, we have decomposed the 9-dimensional space of tensors of rank 2 into the sum of the 3-dimensional space of pseudo-vectors (skew-symmetric tensors of rank 2) and the 6-dimensional space of symmetric tensors of rank 2. The representation in the space of pseudo-vectors is irreducible. In the space of symmetric tensors, the representation can be reduced further. Every symmetric tensor can be represented as the sum of a symmetric tensor with trace zero and a tensor which is multiple of the identity tensor. In fact, we have $c_{ij} = \lambda\delta_{ij} + d_{ij}$, where $\lambda = \frac{1}{3}c_{ii}$ and the symmetric tensor $d_{ij} = c_{ij} - \lambda\delta_{ij}$ has trace zero.

We have thus decomposed the 9-dimensional space of tensors of rank 2 into the sum of invariant subspaces: the 1-dimensional space of tensors of the form $\lambda\delta_{ij}$, *the 3-dimensional space of skew-symmetric tensors, and the 5-dimensional space of symmetric tensors with trace zero.*

The irreducibility of the representation in the first two subspaces is evident. It would not be difficult to show that in the last-named subspace, we have a realization of the irreducible representation with $l=2$. However, this will also follow from the sequel.*

The decomposition into irreducible parts of the tensor representation of rank 3 is more complicated. We shall obtain it in paragraph 3 as a special case of more general considerations.

2. Computation of the weights of the irreducible representations into which a tensor representation is decomposed. In the present paragraph, we shall identify the irreducible representations into which an arbitrary tensor representation can be decomposed. The solution of this problem depends upon the results of the preceding section, dealing with the decomposition into irreducible parts of the product of two irreducible representations.

We know that the tensor representation of rank 1 is itself the irreducible representation of weight $l=1$ (the principal representation).

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* This result also follows from §4. In fact, the tensor representation with $r=2$ is the product of two irreducible representations with $r=1$ (the principal representation). It follows that this representation can be decomposed into irreducible representations with weights $l=0, 1, 2$.

Consider the tensor representation of rank *2*. It is the product of two irreducible representations of weight *1* and hence it can be decomposed into 3 irreducible representations with weights *0*, *1*, and *2*, respectively.

The tensor representation of rank *3* is the product of the tensor representation of rank *2* with the *3*-dimensional irreducible representation. The space R of tensors of rank *2* is decomposable, as pointed out above, into the sum of 3 invariant subspaces (let us denote them by R_0, R_1, and R_2), in which the representation is irreducible, with weights $l = 0, 1, 2$ respectively. Therefore the product $R \times R_1$ of this space with the *3*-dimensional space R_1 can be decomposed into the sum of the invariant subspaces $R_0 \times R_1$, $R_1 \times R_1$, and $R_1 \times R_2$. Since the representation in each of these subspaces is the product of two irreducible representations, we can apply the results of §4. In this way, we infer the following facts.

$R_2 \times R_1$, being the product of representation spaces with weights *2* and *1*, is decomposed into 3 invariant subspaces, in which the representation is irreducible and has weights *3*, *2*, *1*, respectively.

$R_1 \times R_1$ is decomposed into invariant subspaces in which the representation is irreducible and has weights *2*, *1*, *0*, respectively.

$R_1 \times R_0$ is an invariant subspace in which the representation is irreducible and has weight *1*.

Assembling these results, we see that the tensor representation of rank *3* can be decomposed into the sum of 1 irreducible representation of weight *0*, 3 irreducible representations of weight *1*, 2 irreducible representations of weight *2*, and 1 irreducible representation of weight *3*.

Since the tensor representation of rank *4* is the product of representations of rank *3* and rank *1*, we can find the irreducible representations which make up this tensor representation, by using the decomposition already found for rank *3*. That is, we must find the irreducible representations into which each of the summands of the rank *3* decomposition, when multiplied by the rank *1* representation, is decomposed, and then add the results.

The computation is conveniently carried out with the aid of the following table.

Table I

Weight of the representation / rank of the tensor	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$
$r=0$	1					
$r=1$		1				
$r=2$	1	1	1			
$r=3$	1	3	2	1		
$r=4$	3	6	6	3	1	
$r=5$	6	15	15	10	4	1

Each row of the table corresponds to a tensor representation of given rank. The columns correspond to the irreducible representations of different weights. From the foregoing, we infer the following simple rule for filling in the rows of the table. In each place of the r-th row, beginning with the second column, we enter the sum of the numbers in the $(r-1)$st row which lie just to the left, directly above, and just to the right of this place.

In fact, the representation of weight l in the decomposition of the tensor representation of rank r appears when the representations of weights $l-1$, l, and $l+1$ in the tensor representation of rank $r-1$ are multiplied by the representation of weight 1.

In the first column of the r-th row, we enter the number appearing in the 2nd column of the $(r-1)$st row, since the decomposition of the product of a representation of weight 0 with a representation of weight 1 is again simply a representation of weight 1.

In the first column of the table, we find representations with weight $l=0$. A quantity which is transformed with respect to such a transformation is either a scalar or a pseudo-scalar, depending upon how it behaves under reflections. Since a tensor of rank r is multiplied by $(-1)^r$ upon reflection in the origin, the elements in the first column are scalars for even r and pseudo-scalars for odd r. Quantities which are transformed with respect to a representation of weight $l=1$ are vectors if their components change sign under reflections and are pseudo-vectors in the opposite case. Hence the representations of the second column describe pseudo-vectors for even r and vectors for odd r.

More generally, representations of the l-th column describe l-vectors if $r+l$ is even and the corresponding pseudo-quantities if $r+l$ is odd.

3. Decomposition of a tensor representation into multiples of irreducible representations. Tensors of rank 3.* In paragraph 2, we saw that an irreducible representation with a given weight l appears, generally speaking, several times in the decomposition of a tensor representation into irreducible parts. In this paragraph, we indicate a process of decomposition of a tensor representation into representations which are multiples of a single irreducible representation (the definition of such representations is given in §2).

We shall carry out this decomposition by the method described in §2. We know from this section that vectors which are transformed with respect to an irreducible representation of weight l must satisfy the equation

$$H^2 f - l(l+1) f = 0, \tag{6}$$

where $H^2 = H_1^2 + H_2^2 + H_3^2 = -(A_1^2 + A_2^2 + A_3^2)$ and the A_k are the transformations which in the representation correspond to infinitesimal rotations about the co-ordinate axes. When we compute H^2 for the tensor representation under study and assign to l in equation (6) all values appearing in the r-th row of Table I, we obtain a certain set of equations. The solutions of each of these equations form an invariant subspace, in which the representation is a multiple of the irreducible representation with the corresponding value of l.

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* For a more detailed discussion of the decomposition of a tensor representation into irreducible parts, see H. Weyl, *The classical groups*, Princeton University Press, 1939.

We shall now find the form of the equations which define these invariant subspaces, that is, we shall compute the transformation $H^2 = -(A_1^2 + A_2^2 + A_3^2)$ for a given tensor representation. We must first of all determine the transformations A_1, A_2, and A_3. The tensor representation of rank r is the product of r 3-dimensional representations. In §4, we showed that the transformation A_k which corresponds to an infinitesimal rotation about the k-th co-ordinate axis, has the form

$$A_k(efg\cdots) = A_kefg + eA_kfg + efA_kg + \cdots,$$

where e, f, g are vectors in the spaces in which the factor-representations act. Therefore, in order to determine how the transformation A_k acts on the tensor $a_{i_1i_2\ldots i_r}$, we must apply this transformation in turn to each of the indices of the tensor, just as if it were a 3-dimensional vector, meanwhile keeping the other indices unaltered. We then add the results. It is hence sufficient to know how the A_k act on vectors, that is, what are the matrices of these transformations in 3-dimensional space. But this is well known. Choosing as a basis in 3-dimensional space the usual co-ordinates, then it is easy to see that the matrices A_1, A_2, and A_3 have, in this basis, the form

$$A_1 = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{matrix} \right\|, \quad A_2 = \left\| \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{matrix} \right\|, \quad A_3 = \left\| \begin{matrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|.$$

A brief formula for the A_k can be obtained as follows. Let A_k be denoted by A_{st}, where the triple of numbers k, s, t is obtained from the triple of numbers 1, 2, 3 by cyclic permutation. Then we can write

$$A_{st} = \| \alpha^{st}_{ij} \|, \quad \text{where} \quad \alpha^{st}_{ij} = -\delta_{is}\delta_{jt} + \delta_{it}\delta_{js}.$$

Thus the transformation A_{st} acts on the vector a_1, a_2, a_3 according to the formula

$$A_{st}a_j = \alpha^{st}_{ij}a_j = -\delta_{is}\delta_{jt}a_j + \delta_{it}\delta_{js}a_j = -\delta_{is}a_t + \delta_{it}a_s. \tag{7}$$

Consider a tensor of rank r, $a_{i_1i_2\ldots i_r}$. Applying the transformation A_{st} in turn to each of the indices and adding the results, we obtain

$$A_{st}a_{i_1i_2\ldots i_r} = \sum_{p=1}^{r} -\delta_{i_ps}a_{i_1i_2\ldots i_{p-1}ti_{p+1}\ldots i_r} + \delta_{i_pt}a_{i_1i_2\ldots i_{p-1}si_{p+1}\ldots i_r}. \tag{8}$$

Since we require the transformation $(A_{st})^2$, we apply the transformation A_{st} to the tensor written on the right-hand side of (8). this gives us

$$\begin{aligned}(A_{st})^2a_{i_1i_2\ldots i_r} = \sum_{q=1}^{r} -\delta_{i_qs}\Bigg[\sum_{\substack{p=1\\p\neq q}}^{r}(&-\delta_{i_ps}a_{i_1\ldots i_{q-1}ti_{q+1}\ldots i_{p-1}ti_{p+1}\ldots i_r} + \\ &+\delta_{i_pt}a_{i_1i_2\ldots i_{q-1}ti_{q+1}\ldots i_{p-1}si_{p+1}\ldots i_r}) - \delta_{ts}a_{i_1\ldots i_{q-1}ti_{q+1}\ldots i_r} + \\ &+\delta_{tt}a_{i_1\ldots i_{q-1}si_{q+1}\ldots i_r}\Bigg] + \delta_{i_qt}\Bigg[\sum_{\substack{p=1\\p\neq q}}^{r}(-\delta_{i_ps}a_{i_1\ldots i_{q-1}si_{q+1}\ldots i_{p-1}ti_{p+1}\ldots i_r} + \\ &+\delta_{i_pt}a_{i_1\ldots i_{q-1}si_{q+1}\ldots i_{p-1}si_{p+1}\ldots i_r}) - \delta_{ss}a_{i_1\ldots i_{q-1}ti_{q+1}\ldots i_r} + \delta_{st}a_{i_1\ldots i_{q-1}si_{q+1}\ldots i_r}\Bigg].\end{aligned} \tag{9}$$

For substitution in (6), we must find the transformation $H^2 = -A_{23}^2 - A_{31}^2 - A_{12}^2$.

But formula (8) implies at once that $A_{ss} = 0$ and that $A_{ts} = -A_{st}$, so that $(A_{ts})^2 = (A_{st})^2$. Therefore, when we add up $(A_{st})^2$ for all possible values of s and t from 1 to 3, we obtain the transformation $-2H^2$. Carrying out this addition in formula (9) and dividing the result by 2, we have

$$-H^2 = \sum_{q=1}^{r} \sum_{\substack{p=1\\ p\neq q}}^{r} (\delta_{i_p i_q} a_{i_1\dots i_{q-1} t i_{q+1}\dots i_{p-1} t i_{p+1}\dots i_r} - a_{i_1\dots i_{q-1} i_p i_{q+1}\dots i_{p-1} i_q i_{p+1}\dots i_r}) - \\ - 2r a_{i_1 i_2\dots i_r}. \qquad (10)$$

We thus see that *the operator* $-H^2$ *is the sum of all possible traces of the tensor* $a_{i_1 i_2\dots i_r}$*, multiplied by the identity tensor with corresponding indices, minus the sum of the tensors obtained from* $a_{i_1 i_2\dots i_r}$ *by all possible transpositions of two indices, and minus the tensor* $a_{i_1 i_2\dots i_r}$ *itself, multiplied by 2r.*

The equation $(-H^2 + l(l+1))a = 0$ therefore has the form

$$\sum_{\substack{p,q=1\\ p\neq q}}^{r} \delta_{i_p i_q} a_{i_1 i_2\dots i_{q-1} t i_{q+1}\dots i_{p-1} t i_{p+1}\dots i_r} - a_{i_1\dots i_{q-1} i_p i_{q+1}\dots i_{p-1} i_q i_{p+1}\dots i_r} + \\ + [l(l+1) - 2r]\, a_{i_1 i_2\dots i_r} = 0. \qquad (11)$$

The solutions of this equation for each l form an invariant subspace, in which the representation acts as a multiple of the irreducible representation with the corresponding value of l.

For tensors of rank 2, this method of decomposition leads once again to the decomposition developed in paragraph 2, where this space of tensors is decomposed into parts which are skew-symmetric, symmetric with trace zero, and multiples of the identity.

We examine in more detail the decomposition of tensors of rank 3. According to the table in paragraph 2, we must obtain 1 representation of weight 0 (pseudoscalars), 3 of weight 1 (vectors), 2 of weight 2, and 1 of weight 3.

For the tensor a_{ijk}, equation (11) assumes the form, after division by 2,

$$\delta_{ij} a_{ssk} + \delta_{jk} a_{iss} + \delta_{ki} a_{sjs} - a_{jik} - a_{ikj} - a_{kji} + \left[\frac{l(l+1)}{2} - 3\right] a_{ijk} = 0. \qquad (12)$$

We must substitute $l = 0, 1, 2, 3$ in (12).

Form the trace of the left side with respect to the indices i and j. Then the 2nd and 3rd summands cancel the 5th and 6th, respectively, and we obtain

$$2a_{ssk} + \left[\frac{l(l+1)}{2} - 3\right] a_{ssk} = 0,$$

that is

$$\left[\frac{l(l+1)}{2} - 1\right] a_{ssk} = 0.$$

For $l(l+1) \neq 2$, that is, $l \neq 1$, we thus obtain $a_{ssk} = 0$ and similarly $a_{sjs} = a_{iss} = 0$. Therefore, *for tensors of rank 3, the subspaces in which irreducible representations or multiples of irreducible representations with weights* $l \neq 1$ *act, consist of tensors for which all traces are equal to zero.*

We set $l = 0$. Noting that all traces of a_{ijk} are equal to zero, we obtain the equation

$$a_{ijk} = -\frac{a_{jik} + a_{ikj} + a_{kji}}{3}. \tag{13}$$

Under cyclic permutation of indices, the last three summands go into each other, and their sum does not change. Consequently, the tensor a_{ijk} is also invariant under cyclic permutations of indices. Thus

$$a_{ijk} = a_{jki} = a_{kij}.$$

Interchanging the indices i and j in the equation (13), we show in just the same way that

$$a_{jik} = a_{kij} = a_{ikj}.$$

Turning once again to the original equation (13), we find, finally, that

$$a_{ijk} = -a_{jik}.$$

Assembling all of these results, we see that a solution of equation (13) is a skew-symmetric tensor of rank *3*.

Set $l = 1$. Then the equation assumes the form

$$\delta_{ij}a_{ssk} + \delta_{jk}a_{iss} + \delta_{ki}a_{sjs} - a_{jik} - a_{ikj} - a_{kji} - 2a_{ijk} = 0. \tag{14}$$

Substituting the tensor $\delta_{ij}x_k$ in this equation (x_k is an arbitrary vector), we obtain

$$\delta_{ij}\delta_{ss}x_k + \delta_{jk}\delta_{is}x_s + \delta_{ki}\delta_{sj}x_s - \delta_{ij}x_k - \delta_{ik}x_j - \delta_{kj}x_i - 2\delta_{ij}x_k = 0. \tag{15}$$

This shows that the tensor $\delta_{ij}x_k$ satisfies equation (14). The tensors $\delta_{ik}y_j$ and $\delta_{jk}z_i$ also satisfy the equation (14), and therefore also the tensor

$$a_{ijk} = \delta_{ij}x_k + \delta_{ik}y_j + \delta_{jk}z_i. \tag{16}$$

We found in paragraph 2 that in the decomposition of tensor space of rank *3*, the irreducible representation of weight *1* appears three times. It follows from this that the solutions of equation (14) must form a *9*-dimensional space. But the 3 vectors x_k, y_j, and z_i together have 9 components. Let us show that if the tensor (16) is zero, then all 9 of these components must be zero. This implies, of course, that the space of all tensors of the form (16) is actually *9*-dimensional. Setting $a_{ijk} = 0$ in formula (16) and taking all traces of this equation, we obtain 9 equations in the unknowns x_k, y_j, and z_i, having only the solution zero.

Set $l = 2$. Since all traces of a solution of this equation have been proved equal to zero, the equation has the form

$$a_{jik} + a_{ikj} + a_{kji} = 0. \tag{17}$$

The dimension of the space of solutions of this equation must be equal to *10*, in accordance with the results of paragraph 2. We shall show in the sequal how to decompose the representation in this space into 2 irreducible representations with weight $l = 2$.

Finally, set $l = 3$. Recalling that all traces are equal to zero, we obtain

$$a_{ijk} = \frac{a_{jik} + a_{ikj} + a_{kji}}{3}.$$

Reasoning as in the case $l = 0$, we find that in this case the components of a_{ijk} do not change under any permutation of indices, that is, a_{ijk} is a symmetric tensor of rank *3*. It follows from the results of paragraph 2 that the dimension of the subspace of symmetric tensors of rank 3 is equal to 7.

We have, then, the following final result. *The 27-dimensional space of tensors of rank 3 is decomposed into the sum of the following subspaces, in which representations act which are multiples of irreducible representations: the 1-dimensional space of skew-symmetric tensors or pseudo-vectors, which are transformed by the irreducible representation of weight 0; the 9-dimensional subspace of tensors of the form* $a_{ijk} = \delta_{ij}x_k + \delta_{ik}y_j + \delta_{jk}z_i$, *in which there operates a three times repeated irreducible representation of weight 1; the 10-dimensional subspace of tensors which satisfy the equation* $a_{jik} + a_{ikj} + a_{kji} = 0$, *in which there operates a twice repeated irreducible representation of weight 2; and finally, the 7-dimensional subspace of symmetric tensors in which all traces are zero. In this last subspace there acts the irreducible representation of weight 3. It follows from the general results of §3 that this decomposition into representations which are multiples of irreducible representations is unique.*

In order to decompose the tensor representation of rank 3 into irreducible representations, it is necessary to decompose into irreducible parts those of the representations already obtained which are multiples of irreducible representations. This can be done in various ways. We indicate a few of them.

The 9-dimensional space of tensors of the form $a_{ijk} = \delta_{ij}x_k + \delta_{ik}y_j + \delta_{jk}z_i$ can be decomposed in a natural way into 3 3-dimensional subspaces: $a^{(1)}_{ijk} = \delta_{ij}x_k$, $a^{(2)}_{ijk} = \delta_{ik}y_j$, and $a^{(3)}_{ijk} = \delta_{jk}z_i$. In each of these subspaces, the representation is irreducible. This decomposition is of course not unique. Every triple of linearly independent linear combinations of the vectors x_k, y_j and z_i generate such a decomposition.

We next consider irreducible representations which employ tensors satisfying equation (17): $a_{ijk} + a_{jki} + a_{kij} = 0$. As we have seen, the dimensionality of the space of solutions of (17) is 10, and a twice repeated irreducible representation of weight 2 acts in this space. Hence *every* decomposition of this representation must be into irreducible parts, and it suffices to take any decomposition whatsoever of this space into invariant subspaces.

We can do this by carrying out a decomposition of the space with the aid of eigenvalues of a permutation transformation, as described in paragraph 1. Thus, for example, it is possible to decompose the space of tensors satisfying equation (17) into the subspace of tensors which are symmetric and skew-symmetric, respectively, with respect to an arbitrary pair of indices, since among the non-zero solutions of (17) there are tensors of both kinds.

One can also employ, to find such a decomposition, the cylic permutations of all 3 indices which carries ijk into jki. It is clear that $s^3 = s_0$, and hence the eigenvalues of the corresponding transformation S are all cube roots of unity:

$$1, \quad \varepsilon = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \varepsilon^2 = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

The equation $Sa_{ijk} = a_{ijk}$ together with the equation (17) gives $a_{ijk} = 0$. Hence there remain the 2 equations

$$Sa_{ijk} = a_{jki} = \varepsilon a_{ijk}$$

and

$$Sa_{ijk} = a_{jki} = \varepsilon^2 a_{ijk},$$

which produce a decomposition of our representation into irreducible parts. One could deal in just the same way with any other permutation transformation.

We have thus found that *the decomposition of the tensor representation of rank 3 is not unique. Every decomposition into irreducible parts of the representations which are multiples of irreducible representations gives a certain decomposition of the entire tensor representation.*

§6. Spinors and spinor representations

In the preceding section, we saw that it is possible to realize irreducible representations of the rotation group with arbitrary integer l by the aid of tensor representations. In this section, we consider a different concrete realization of representations of the rotation group; these are the spinor representations, which enable us to realize all irreducible representations of this group without any exceptions. Representations with half-integer l are included, that is, the two-valued representations.

1. Definition of a spinor and of a spinor representation. We begin with the definition of a spinor and of a spinor representation of rank 1. In §2, we took as an example the irreducible representation with $l = 1/2$. In this representation, the rotation g with Euler angles ϕ_1, θ, ϕ_2 was mapped into the complex unitary matrix of order 2

$$\begin{Vmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{Vmatrix}, \tag{1}$$

where

$$\alpha = \pm \cos\frac{\theta}{2} e^{i\frac{\varphi_1+\varphi_2}{2}}, \quad \beta = \mp i \sin\frac{\theta}{2} e^{i\frac{\varphi_2-\varphi_1}{2}}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

This matrix is defined up to its sign. The product of two matrices corresponds to the product of the corresponding rotations.

We now suppose that we have in every system of co-ordinates a pair of complex numbers (a^1, a^2) which are defined up to their sign. We suppose further that in going from one co-ordinate system to another by means of the rotation $g = \|g_{ik}\|$,* the number pair (a^1, a^2) is transformed by the matrix (1) according to the formulas

$$\left.\begin{aligned} a^{1\prime} &= \alpha a^1 + \beta a^2, \\ a^{2\prime} &= -\overline{\beta} a^1 + \overline{\alpha} a^2. \end{aligned}\right\} \tag{2}$$

Such a system of numbers is called a spinor of rank 1 in 3-dimensional Euclidean space. The numbers a^1 and a^2 are called the components of the spinor.

If we think of formula (2) as a formula which transforms a spinor with components $\{a^1, a^2\}$ into another spinor with components $\{a^{1\prime}, a^{2\prime}\}$ the co-ordinate system being fixed, then the transformations (2) produce a 2-dimensional representation of the rotation group. This representation is called the spinor representation of rank 1. It follows from what has been said above that it is the irreducible representation of weight $1/2$. For further use, it is convenient to rewrite the matrix

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* That is, by the transformation of co-ordinates given by the formulas $x_i' = \Sigma g_{ik} x_k$.

$$\begin{Vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{Vmatrix} \quad \text{as} \quad \begin{Vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{Vmatrix},$$

that is, to set $a = a_{11}$, $\beta = a_{12}$, $-\bar{\beta} = a_{21}$, $\bar{a} = a_{22}$. Then the formulas according to which the components of a spinor are transformed assume the form

$$a^{i'} = \sum_{k=1}^{2} \alpha_{ik} a^k, \qquad \mathrm{Det}\,|\alpha_{ik}| = 1. \tag{2'}$$

Suppose now that in every orthogonal system of co-ordinates, we have a set of 2^r *complex numbers* $a^{\lambda_1\lambda_2\ldots\lambda_r}$ $(\lambda_1, \lambda_2, \cdots, \lambda_r = 1, 2)$, *which are defined up to their sign. Suppose that under a passage from one co-ordinate system to another by the use of the matrix* $\|a_{ik}\|$. *this set of complex numbers is transformed according to the formulas*

$$a'^{\lambda'_1\lambda'_2\ldots\lambda'_r} = \sum\sum \cdots \sum \alpha_{\lambda'_1\lambda_1}\alpha_{\lambda'_2\lambda_2} \cdots \alpha_{\lambda'_r\lambda_r} a^{\lambda_1\lambda_2\ldots\lambda_r}, \tag{3}$$

where $a_{11} = \bar{a}_{22} = a$, $a_{12} = -\bar{a}_{21} = \beta$. *Such a system of numbers is called a contravariant spinor of rank* r *in 3-dimensional Euclidean space.*

The set of all spinors of rank r, described in a certain system of co-ordinates, forms a linear space of dimension 2^r.

The formula (3), giving the transformation of spinors, also defines a representation of the rotation group in the space of spinors of rank r. This representation is known as the spinor representation of rank r. It is clear from formula (3) that every component of a spinor of rank r is transformed as the product of the components of r spinors of rank *1*. Hence the spinor representation of rank r is the product of r irreducible representations of weight *1/2*. This is in complete analogy with the fact that a tensor representation is a product of irreducible representations of weight *1*.

2. Symmetric spinors. Existence of irreducible representations for every half-integer weight l. In this paragraph, we shall prove that for every integer and half-integer value of l, there exists a subspace of spinors which under the rotation group is transformed with respect to the irreducible representation of weight l. The subspaces of symmetric spinors serve in this capacity.

A spinor $a^{i_1i_2\ldots i_r}$ of rank r is called *symmetric*, if its components are not altered under any permutation of the indices $i_1 i_2 \cdots i_r$. Since the indices i_a can assume only the values *1* and *2*, it is clear that by suitable permutation of indices, an arbitrary component of a symmetric spinor can be brought into coincidence with one of the following:

$$a^{\overbrace{11\ldots1}^{r}}, \quad \ldots, \quad a^{\overbrace{11\ldots1}^{r-k},\, \overbrace{22\ldots2}^{k}}, \quad \ldots, \quad a^{\overbrace{22\ldots2}^{r}}.$$

It is evident from this that the symmetric spinors of rank r form an $(r+1)$-dimensional subspace in the space of all spinors. It is easy to show that this subspace is invariant under the spinor representation. In fact, it is clear from the formula

$$a'^{i'_1i'_2\ldots i'_r} = \sum_{i_1=1}^{2}\sum_{i_2=1}^{2} \cdots \sum_{i_r=1}^{2} \alpha_{i'_1i_1}\alpha_{i'_2i_2} \cdots \alpha_{i'_ri_r} a^{i_1i_2\ldots i_r},$$

according to which spinors of rank r are transformed, that one and the same matrix $\|a_{ik}\|$ operates on each of the indices. Hence the symmetry of the spinor is not destroyed under the transformation.

Let us show that the representation is actually irreducible in the space of symmetric spinors of arbitrary rank r. From this point onward, it will be convenient to write r as $2l$, where l is an integer or a half-integer. In order to prove the irreducibility of the representation in the space of symmetric spinors of rank $2l$, it suffices to prove that the transformation H_3 in this space has $2l+1$ different eigenvalues (that is, as many as the dimension of the space). For this purpose, we must determine the form of the transformation H_3 in the spinor representation. The spinor representation of rank 1 is the irreducible representation of weight $1/2$, and for this representation, the matrix $H_3 = iA_3$ has the form

$$\begin{Vmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{Vmatrix}$$

as was shown in section 2, paragraph 5. Since the spinor representation of rank r is the product of r representations of weight $1/2$, we compute $H_3 a^{i_1 i_2 \cdots i_r}$ by applying this matrix in turn to each of the indices i_a, leaving the others fixed, and then adding the results (see §4, paragraph 2). We obtain in this way

$$H_3 a^{i_1 i_2 \ldots i_r} = \left(-\frac{p_1}{2} + \frac{p_2}{2} \right) a^{i_1 i_2 \ldots i_r}.$$

Here p_1 is the number of 1's among the indices i_a and p_2 is the number os 2's. It is clear from this formula that the spinor $a^{i_1 i_2 \cdots i_r}$ in which the only non-zero components are those with p_1^0 1's and p_2^0 2's is an eigenvector of H_3 with eigenvalues $\frac{1}{2}(p_2^0 - p_1^0)$.

Let f_m be the symmetric spinor of rank $2l$ in which $a^{\overbrace{11 \ldots 1}^{l-m}, \overbrace{22 \ldots 2}^{l+m}} = 1$, and all components different from this one are equal to zero. Then

$$H_3 f_m = \left[-\frac{1}{2}(l-m) + \frac{1}{2}(l+m) \right] f_m = m f_m.$$

Thus f_m is an eigenvector with corresponding eigenvalue m for the transformation H_3. Since m can assume the values $-l, -l+1, \cdots, l$, we obtain in this way $2l+1$ eigenvectors of H_3 in the space of symmetric spinors, with different eigenvalues. By this, we have proved that the representation is irreducible in this space. The spinors differ only by constant multiples from the elements of the canonical basis for this representation.

We have thus shown that all irreducible representations of the rotation group can be realized in spaces of symmetric spinors. In the space of symmetric spinors of rank $2l$, we find the irreducible representation of weight l, since the dimension of the space is $2l+1$. It follows that the symmetric spinors of even rank produce irreducible representations of integer weight, which we have already encountered, while the symmetric spinors of odd rank produce irreducible representations with half-integer weight. In particular, the symmetric spinors of rank 2 produce the representation with $l = 1$. It follows that there is a mapping of the space of symmetric spinors of rank 2 onto the space of vectors of dimension 3, such that corresponding elements are transformed in just the same way under rotations. To describe this mapping, we note that if a^{ik} is a symmetric spinor, then its components a^{11}, $\sqrt{2a}^{12}$, a^{22} are its co-ordinates in the canonical basis. Since the components in the canonical basis of a vector a_x, a_y, a_z are $\frac{a_x + ia_y}{\sqrt{2}}$, a_z, $\frac{-a_x + ia_y}{\sqrt{2}}$ (see § 2, paragraph 5), the mapping in question is defined by

$$a_x = +\frac{1}{\sqrt{2}}(a^{11} - a^{22}),$$

$$a_y = -\frac{i}{\sqrt{2}}(a^{11} + a^{22}),$$

$$a_z = \sqrt{2}\, a^{12}.$$

3. **The principal operations on spinors.** At the present time, spinors and spinor representation are widely applied in theoretical physics. In connection with this, we discuss certain operations of spinor algebra, which are analogous to the corresponding operations for tensors (see § 5).

We begin with the operation of contraction or formation of the trace. Consider first two spinors of rank 1, $\{a^1, a^2\}$ and $\{b^1, b^2\}$, and form the determinant

$$a^1 b^2 - a^2 b^1 \tag{4}$$

from the components of these two spinors. Under transformation of co-ordinates, this determinant is multiplied by the determinant of the matrix $\|a_{ik}\|$. In view of the fact that $a_{11}a_{22} - a_{12}a_{21} = |\alpha|^2 + |\beta|^2 = 1$, the determinant (4) does not change under passage to another co-ordinate system. Therefore the bilinear form $a^1 b^2 - a^2 b^1$, formed from the components of two spinors of rank 1, is a scalar.

We introduce next the matrix

$$\| \varepsilon_{\alpha\beta} \| = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}. \tag{5}$$

Using this matrix, we can write this scalar as

$$\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \varepsilon_{\alpha\beta} a^{\alpha} b^{\beta}.$$

We define the number b_α as follows:

$$b_\alpha = \sum_{\beta=1}^{2} \varepsilon_{\alpha\beta} b^{\beta}. \tag{6}$$

In other words, $b_1 = b^2$, $b_2 = -b^1$. It is clear that the numbers b_1 and b_2 form a spinor if b^1 and b^2 do so. These numbers bear the name of *covariant components* of the spinor $\{b^1, b^2\}$. It is easy to deduce from formulas (4) and (2) that under transformation of co-ordinates, the covariant components of a spinor are transformed with the aid of the matrix which is the complex adjoint of the matrix (1).

Using the covariant components of a spinor, the form (4) can be written as the sum of product $a^1 b_1 + a^2 b_2 = \Sigma_{\lambda=1}^{2} a^{\lambda} b_{\lambda}$. In the sequel, we will omit the summation sign in formulas where the summation is carried out with respect to an upper and a lower index, that is, we shall write formula (4) simply as $a^{\lambda} b_{\lambda}$.

We now consider an arbitrary spinor $a^{\lambda_1 \lambda_2 \ldots \lambda_r}$, and form the following system of numbers from its components:

$$b^{\lambda_3 \lambda_4 \ldots \lambda_r} = \varepsilon_{\alpha\beta} a^{\alpha\beta\lambda_3 \ldots \lambda_r}. \tag{7}$$

Expression (7) is called the trace of the spinor $a^{\lambda_1 \ldots \lambda_r}$ with respect to the first two indices. It follows from formula (4) that the trace of a spinor of rank r is a spinor of rank $r-2$. In fact, a spinor of rank r is transformed like the product of r spinors of rank 1, and consequently

$$b'^{\lambda_3'\dots\lambda_r'} = \alpha_{\lambda_3'\lambda_3}\alpha_{\lambda_4'\lambda_4}\ \dots\ \alpha_{\lambda_r'\lambda_r} b^{\lambda_3\dots\lambda_r}.$$

If, as in the case of a spinor of rank 1, we introduce components for the spinor $a^{\lambda_1\lambda_2\dots\lambda_r}$ with one index lowered (covariant in one index and contravariant in the rest), as follows:

$$a_{\lambda_1}^{\lambda_2\dots\lambda_r} = \varepsilon_{\lambda_1\alpha} a^{\alpha\lambda_2\dots\lambda_r},$$

then the trace of the spinor $a^{\lambda_1\lambda_2\dots\lambda_r}$ with respect to the first two indices can be briefly written as

$$\operatorname{Tr} a^{\lambda_1\lambda_2\dots\lambda_r} = a_\beta^{\beta\lambda_3\dots\lambda_r}.$$

The operation of forming the trace, as in the case of tensors, is called contraction of the spinor with respect to the first two indices. The operation of contraction with respect to any pair of distinct indices is defined similarly.

In defining contraction, we encountered the operation of lowering an index, which changes the contravariant components of the spinor $a^{\lambda_1\lambda_2\dots\lambda_r}$ into mixed components (covariant in one and contravariant in the other indices). These components are written as $a_{\lambda_1}^{\lambda_2\dots\lambda_r}$. In the same way, we can lower an arbitrary number of indices and obtain mixed components which are covariant in some indices and contravariant in the others. Under transformation of co-ordinates, the upper indices are transformed by the matrix $\|a_{ik}\|$ and the lower indices by the complex adjoint matrix of $\|a_{ik}\|$. Thus the formulas for the transformation of the component $a_{\sigma\rho\tau\dots}^{\lambda\mu\nu\dots}$ has the form

$$a'^{\lambda'\mu'\nu'\dots}_{\sigma'\rho'\tau'\dots} = \alpha_{\lambda'\lambda}\alpha_{\mu'\mu}\alpha_{\nu'\nu}\ \dots\ \bar{\alpha}_{\sigma'\sigma}\bar{\alpha}_{\rho'\rho}\bar{\alpha}_{\tau'\tau} a^{\lambda\mu\nu\dots}_{\sigma\rho\tau\dots}.$$

The matrix $\|\epsilon^{\alpha\beta}\|$, the inverse of the matrix $\|\epsilon_{\alpha\beta}\|$, on the other hand, "raises" the indices of a spinor. Using it, we can change any mixed components of a spinor into contravariant components, which transform according to formulas (3).

Let $a_{i_{p+1},\dots i_r}^{i_1\dots i_p}$ and $b_{j_{p'+1}\dots j_s}^{j_1\dots j_{p'}}$ be two spinors, each of which is contravariant with respect to part of its indices and covariant with respect to the others. The product of these two spinors is the spinor

$$c^{i_1\dots i_p j_1\dots j_{p'}}_{i_{p+1}\dots i_r j_{p'+1}\dots j_s}$$

of rank $r+s$, whose components are all possible products of the components of the two factors. The fact that this product is again a spinor is proved just as in the case of tensors.

4. Decomposition of a spinor representation into irreducible parts. Suppose now that we have an arbitrary spinor of rank r. We know that irs transformations form a representation of dimension 2^r of the rotation group. In the present paragraph, we shall determine the kind and number of the irreducible parts of this representation.

We can solve this problem in just the same way as in the case of tensors. In fact, we already know that spinors of rank 1 are transformed by the irreducible representation with weight $1/2$. An arbitrary spinor of rank 2 is transformed as the product of two spinors of rank 1. Since each of the factors is transformed according to the irreducible representation of weight $1/2$, the product can be decomposed into the sum of irreducible representations with weights $\frac{1}{2} - \frac{1}{2} = 0$ and $\frac{1}{2} + \frac{1}{2} = 1$. Consider next spinors of rank *3*. They are transformed as the product of spinors of rank *2* and rank *1*. The spinors of rank 2 are not transformed by an irreducible representation of the rotation group. Hence we must consider the representation by spinors of rank *2* as the sum of 2 irreducible representations, and multiply each of its summands by the irreducible representation with weight ½. In doing this, we obtain from the summand with weight *0* a representation of weight ½, and from the summand with weight *1*, a representation with weight ½ and a representation with weight *3/2*. (See table II.) Thus, in decomposing the spinor representation of rank *3* into its irreducible components, we obtain 2 representations of weight ½ and one with weight *3/2*.

We may now continue this process. That is, we consider a spinor of rank r as the product of a spinor of rank $r-1$ and a spinor of rank *l*. Being already in possession of the decomposition into irreducible components of a spinor of rank $r-1$, we can compute the type and number of irreducible components of the representation by spinors of rank r.

Thus, each entry of the following table is found by computing the sum of the numbers immediately to the left and right of it in the preceding row.

Table II

Weight of the representation Rank of the spinor	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
1		1					
2	1		1				
3		2		1			
4	2		3		1		
5		5		4		1	
6	5		9		5		1

§7. Matrix elements of an irreducible representation

(generalized spherical functions)

In §2, we studied the matrices which correspond to infinitesimal rotations around the co-ordinate axes, in an arbitrary irreducible representation of the rotation group. We computed these matrices in a certain special basis. In the present section, we shall find the matrix in this basis which corresponds to an arbitrary rotation g.

1. The operators U_g. Let $g \to T_g$ be an irreducible representation of weight l. The elements T_{mn} of the matrix T_g $(-l \leq m, n \leq l)$ are functions of g, which we must define,

In order to do this, multiply g by an arbitrary rotation g_1. Then $T_{mn}(g)$ goes into a different function of g, which is equal to $T_{mn}(gg_1)$. This transformation of the functions T_{mn} depends upon g_1. Denoting it by U_{g_1}, we can write

$$U_{g_1} T_{mn}(g) = T_{mn}(gg_1). \tag{1}$$

It is easy to verify that the relation

$$U_{g_2} U_{g_1} = U_{g_2 g_1} \tag{2}$$

holds for the transformation U_{g_1}. In fact,

$$U_{g_2} U_{g_1} T_{mn}(g) = U_{g_2} T_{mn}(gg_1) = T_{mn}(gg_2g_1) = U_{g_2 g_1} T_{mn}(g).$$

We consider more clostly the function $T_{mn}(gg_1)$. By definition, this function is an element of the matrix T_{gg_1}. Since the matrices T_g form a representation of the rotation group, it follows that

$$T_{gg_1} = T_g T_{g_1}.$$

Equating the elements of the matrices on the left and right sides of this equality, we find that

$$T_{mn}(gg_1) = \sum_{s=-l}^{s=l} T_{ms}(g) T_{sn}(g_1). \tag{3}$$

This equality implies that

$$U_{g_1} T_{mn}(g) = \sum_{s=-l}^{s=l} T_{ms}(g) T_{sn}(g_1). \tag{4}$$

We thus see that the transformation U_{g_1} carries an element of the m-th row of the matrix T_g into a linear combination of the elements of the same row of T_g, with coefficients which depend upon g_1.

Consider next the $(2l+1)$-dimensional space R^m of functions of g, generated by the elements of the m-th row of the matrix T_g. This space consists of all linear combinations of the functions

$$T_{mn}(g) \qquad (-l \leq n \leq l).$$

It follows from formulas (2) and (4) that for every m, the transformations U_{g_1} comprise a $(2l+1)$-dimensional representation of the rotation group, in the space R^m.

Formula (4) implies also that the coefficients of the image of the function $T_{mn}(g)$ under the transformation U_{g_1} are equal to $T_{sn}(g_1)$ $(-l \leq s, n \leq l)$. This shows that the matrices of the transformations U_{g_1} coincide with the matrices T_{g_1}.

It follows immediately from these considerations that the representation $g_1 \to U_{g_1}$ in the space R^m is irreducible. Furthermore, the functions $T_{mn}(g)$ $(-l \leq m, n \leq l)$ on which the matrices T_{g_1} operate from a canonical basis in the space R^m. Consequently, the transformations H_+ and H_-, defined for the representation $g_1 \to U_{g_1}$, act on the functions $T_{mn}(g)$ in accordance with the formulas obtained earlier (see formula (19), §2).

With the help of these considerations, we shall find the functions $T_{mn}(g)$ and establish a number of recurrence relations among them.

2. Differential operators corresponding to infinitesimal rotations. We must first find the transformations A_k which correspond to infinitesimal rotations about the co-ordinate axes in the representation described in paragraph 1. Just as was done in §3, we must take g_1 as the rotations through an angle a around a fixed axis, and then expand $U_{g_1}T_{mn}(g) = T_{mn}(gg_1)$ in powers of a.

The computation is very simple if the axis oz is taken as the axis of rotation. Let g be an arbitrary rotation, with Euler angles ϕ_1, θ, ϕ_2. Let g_1 be the rotation through the angle a around the axis oz. Then, as is easily seen, the rotation gg_1 has Euler angles $\phi_1+a,\ \theta,\ \phi_2$.* Consequently, we have

$$T_{mn}(gg_1)=T_{mn}(\varphi_1+\alpha,\ \theta,\ \varphi_2)=T_{mn}(\varphi_1,\ \theta,\ \varphi_2)+\alpha\frac{\partial T_{mn}}{\partial\varphi_1}+\dots,$$

and the transformation A_3 is the differential operator

$$A_3=\frac{\partial}{\partial\varphi_1}. \tag{5}$$

In the general case, the expansion of $T_{mn}(gg_1) = T_{mn}(\phi_1', \theta', \phi_2')$ has the form

$$T_{mn}(\varphi_1',\theta',\varphi_2')=T_{mn}(\varphi_1,\theta,\varphi_2)+\alpha\left[\frac{\partial T_{mn}}{\partial\varphi_1}\frac{d\varphi'}{d\alpha}+\frac{\partial T_{mn}}{\partial\theta}\frac{d\theta'}{d\alpha}+\frac{\partial T_{mn}}{\partial\varphi_2}\frac{d\varphi_2'}{d\alpha}\right]_{\alpha=0}+\dots. \tag{6}$$

We next determine the numbers $\left.\frac{d\varphi_1'}{d\alpha}\right|_{\alpha=0}$, $\left.\frac{d\theta'}{d\alpha}\right|_{\alpha=0}$ and $\left.\frac{d\varphi_2'}{d\alpha}\right|_{\alpha=0}$ for the case in which g_1 is the rotation through the angle a around the axis ox.

In order to do this, we consider the matrix of the rotation g itself as a function of its Euler angles. As was shown in §1, it has the form

$$g(\varphi_1,\theta,\varphi_2)=\|g_{ik}(\varphi_1,\theta,\varphi_2)\|=$$

$$=\left\|\begin{matrix}\cos\varphi_2\cos\varphi_1-\cos\theta\sin\varphi_2\sin\varphi_1, & -\cos\varphi_2\sin\varphi_1-\cos\theta\sin\varphi_2\cos\varphi_1, & \sin\varphi_2\sin\theta\\ \sin\varphi_2\cos\varphi_1+\cos\theta\cos\varphi_2\sin\varphi_1, & -\sin\varphi_2\sin\varphi_1+\cos\theta\cos\varphi_2\cos\varphi_1, & -\cos\varphi_2\sin\theta\\ \sin\varphi_1\sin\theta, & \cos\varphi_1\sin\theta. & \cos\theta\end{matrix}\right\| \tag{7}$$

The matrix of the rotation gg_1 is given by certain values of Euler's angles: ϕ_1', θ', ϕ_2'. These values depend upon the rotation angle a and for $a=0$, they are equal to ϕ_1, θ, ϕ_2. Expanding the matrix gg_1 in powers of a, we obtain

$$gg_1=\|g_{ik}(\varphi_1,\theta,\varphi_2)\|+\alpha\left\|\frac{\partial g_{ik}}{\partial\varphi_1}\frac{d\varphi_1'}{d\alpha}+\frac{\partial g_{ik}}{\partial\theta}\frac{d\theta'}{d\alpha}+\frac{\partial g_{ik}}{\partial\varphi_2}\frac{d\varphi_2'}{d\alpha}\right\|_{\alpha=0}+\dots \tag{8}$$

On the other hand, since g_1 is the rotation through the angle a around the axis ox, its matrix is equal to

$$g_1=\left\|\begin{matrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{matrix}\right\|=\left\|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\right\|+\alpha\left\|\begin{matrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{matrix}\right\|+\dots.$$

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* Translator's note. The authors err once again, as a result of their original error in formula (9) of §1. Actually, gg_1 has Euler angles ϕ_1, θ, ϕ_2+a. However, if one reverses the definitions of ϕ_1 and ϕ_2, as suggested in the translator's note in §1, the error disappears, and the assertions of the present paragraph are all correct.

Consequently

$$gg_1 = \| g_{ik}(\varphi_1, \theta, \varphi_2) \| + \alpha \begin{Vmatrix} 0 & g_{13} & -g_{12} \\ 0 & g_{23} & -g_{22} \\ 0 & g_{33} & -g_{32} \end{Vmatrix} + \cdots. \tag{9}$$

Equating the expressions (8) and (9) for the matrix gg_1 and equating the coefficients of a in these two expressions, we obtain equations from which

$$\left.\frac{d\varphi_1'}{d\alpha}\right|_{\alpha=0}, \quad \left.\frac{d\theta'}{d\alpha}\right|_{\alpha=0} \text{ and } \left.\frac{d\varphi_2'}{d\alpha}\right|_{\alpha=0}$$

are defined.

It is sufficient for our purposes to take the three simplest elements of the matrices by which a is multiplied in formulas (8) and (9). We use the lower right, the lower left, and the upper right elements. Taking the corresponding expressions for g_{ik} from formula (7) and differentiating these, we obtain the equations

$$-\sin\theta \left.\frac{d\theta'}{d\alpha}\right|_{\alpha=0} = -\cos\varphi_1 \sin\theta,$$

$$\cos\varphi_1 \sin\theta \left.\frac{d\varphi_1'}{d\alpha}\right|_{\alpha=0} + \sin\varphi_1 \cos\theta \left.\frac{d\theta'}{d\alpha}\right|_{\alpha=0} = 0,$$

$$\cos\varphi_2 \sin\theta \left.\frac{d\varphi_2'}{d\alpha}\right|_{\alpha=0} + \sin\varphi_2 \cos\theta \left.\frac{d\theta'}{d\alpha}\right|_{\alpha=0} = \cos\varphi_2 \sin\varphi_1 + \cos\theta \sin\varphi_2 \cos\varphi_1.$$

From these equations, we infer that

$$\left.\frac{d\theta'}{d\alpha}\right|_{\alpha=0} = \cos\varphi_1, \quad \left.\frac{d\varphi_1'}{d\alpha}\right|_{\alpha=0} = -\sin\varphi_1 \operatorname{ctg}\theta, \quad \left.\frac{d\varphi_2'}{d\alpha}\right|_{\alpha=0} = \frac{\sin\varphi_1}{\sin\theta}.$$

Substituting these expressions in formula (6), we find the differential operator which corresponds to an infinitesimal rotation about the axis ox:

$$A_1 = -\operatorname{ctg}\theta \sin\varphi_1 \frac{\partial}{\partial\varphi_1} + \frac{\sin\varphi_1}{\sin\theta} \frac{\partial}{\partial\varphi_2} + \cos\varphi_1 \frac{\partial}{\partial\theta}. \tag{10}$$

The operator A_2 is computed in much the same way. It has the form

$$A_2 = -\operatorname{ctg}\theta \cos\varphi_1 \frac{\partial}{\partial\varphi_1} + \frac{\cos\varphi_1}{\sin\theta} \frac{\partial}{\partial\varphi_2} - \sin\varphi_1 \frac{\partial}{\partial\theta}. \tag{11}$$

The operators H_+, H_-, and H_3, which are more convenient to use for further calculations, can now be easily determined:

$$\begin{aligned} H_+ &= H_1 + iH_2 = iA_1 - A_2 = e^{-i\varphi_1}\left(\operatorname{ctg}\theta \frac{\partial}{\partial\varphi_1} - \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi_2} + i\frac{\partial}{\partial\theta}\right), \\ H_- &= H_1 - iH_2 = iA_1 + A_2 = e^{i\varphi_1}\left(-\operatorname{ctg}\theta \frac{\partial}{\partial\varphi_1} + \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi_2} + i\frac{\partial}{\partial\theta}\right), \\ H_3 &= iA_3 = i\frac{\partial}{\partial\varphi_1}. \end{aligned} \tag{12}$$

We produce the operator $H^2 = H_1^2 + H_2^2 + H_3^2$ from the above. We can now set down the differential equation satisfied by all of the functions $T_{mn}(\phi_1, \theta, \phi_2)$, just as was done in §3 for spherical functions:

$$\frac{\partial^2 U}{\partial \theta^2} + \operatorname{ctg} \theta \frac{\partial U}{\partial \theta} + \frac{1}{\sin^2 \theta}\left(\frac{\partial^2 U}{\partial \varphi_1^2} - 2 \cos \theta \frac{\partial^2 U}{\partial \varphi_1 \partial \varphi_2} + \frac{\partial^2 U}{\partial \varphi_2^2} \right) + l(l+1) U = 0. \quad (13)$$

We can set down in explicit form the solutions of this equation by the same method as was used in §3 to find the spherical functions $Y_l^m(\phi, \theta)$.

3. **Dependence of the matrix elements on Euler's angles ϕ_1 and ϕ_2.** The functions $T_{mn}(\phi_1, \theta, \phi_2)$ depend upon the variables ϕ_1 and ϕ_2 in an extremely simple way. In fact, we know that an arbitrary rotation can be written as the product of 3 particular rotations: a rotation through the angle ϕ_1 around the axis oz, then a rotation through the angle θ around the axis ox, and finally, a rotation through the angle ϕ_2 around the axis oz. In the representation which we are now studying, let T_{ϕ_1} and T_{ϕ_2} be the matrices corresponding to rotations of amount ϕ_1 and ϕ_2, respectively, about the axis oz. Let T_θ be the matrix which corresponds to a rotation of amount θ around the axis ox. Then, as shown in §1, the equality

$$T_g = T_{\phi_2} T_\theta T_{\phi_1} \quad (14)$$

holds. However, we have already found the matrix which under the irreducible representation of weight l corresponds to the rotation through the angle ϕ around the axis oz. As shown in §2, it has the form

$$T_\varphi = \begin{Vmatrix} e^{il\varphi} & 0 & 0 \dots & 0 \\ 0 & e^{i(l-1)\varphi} & 0 \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots & e^{-il\varphi} \end{Vmatrix}. \quad (14')$$

(See the discussion in paragraph 5 of §2. We recall that the rows and columns of this matrix are numbered from $-l$ to l, so that at the m-th row and m-column, we have the entry $e^{-im\phi}$.)

We replace the matrices T_{ϕ_2} and T_{ϕ_1} in (14) by their expressions according to (14') and then carry out the indicated matrix multiplication. This yields

$$T_{mn}(\varphi_1, \theta. \varphi_2) = e^{-im\varphi_2} u_{mn}(\theta) e^{-in\varphi_1}, \quad (15)$$

where $u_{mn}(\theta)$ denotes the entry in the m-th row and n-th column of the matrix T_θ. It therefore remains only to compute the functions $u_{mn}(\theta)$.

We use formula (15) for $T_{mn}(\phi_1, \theta, \phi_2)$ and substitute into the differential equation (13), which is satisfied by this function. Cancelling the common factor $e^{-im\phi_2} e^{-in\phi_1}$, we obtain an ordinary differential equation for the functions $u_{mn}(\theta)$, namely,

$$\frac{d^2 u}{d\theta^2} + \operatorname{ctg} \theta \frac{du}{d\theta} + \left[l(l+1) - \frac{n^2 - 2mn \cos \theta + m^2}{\sin^2 \theta} \right] u = 0.$$

By the change of variables

$$\tau = \sin^2 \frac{\theta}{2},$$

$$u(\theta) = \tau^{\frac{|m-n|}{2}} (1 - \tau)^{\frac{|m+n|}{2}} v(\tau)$$

this equation is transformed into the hypergeometric equation

$$\tau(1-\tau)\frac{d^2v}{d\tau^2}+[c-(a+b+1)\tau]\frac{dv}{d\tau}-abv(\tau)=0,$$

where

$$\begin{aligned} a &= l+1+\frac{1}{2}(|m-n|+|m+n|),\\ b &= -l+\frac{1}{2}(|m-n|+|m+n|),\\ c &= |m-n|+1. \end{aligned}$$

We shall determine the functions $u_{mn}(\theta)$ explicitly in the next paragraph.

4. Generalized spherical functions. For every value of m, the functions $T_{mn}(\phi_1, \theta, \phi_2)$ are eigenfunctions of the operator H_3, with corresponding eigenvalue n. In particular, the functions $T_{ml}(\phi_1, \theta, \phi_2)$ correspond to the maximum eigenvalue l. As shown in §2, paragraph 3, it follows that these functions must satisfy the equation $H_+T_{ml}(\phi_1, \theta, \phi_2) = 0$, or,

$$e^{-i\varphi_1}\left(\operatorname{ctg}\theta\frac{\partial T_{ml}}{\partial\varphi_1}-\frac{1}{\sin\theta}\frac{\partial T_{ml}}{\partial\varphi_2}+i\frac{\partial T_{ml}}{\partial\theta}\right)=0.$$

Substituting $T_{ml}(\phi_1, \theta, \phi_2) = e^{-im\phi_2}u_{ml}(\theta)e^{-il\phi_1}$ in this equation and cancelling out the common factor $e^{-im\phi_2}e^{-i(l+1)\phi_1}$, we obtain an ordinary differential equation of the first order for $u_{ml}(\theta)$:

$$\frac{du_{ml}(\theta)}{d\theta}+\frac{m-l\cos\theta}{\sin\theta}u_{ml}(\theta)=0. \tag{16}$$

The general solution of this equation is

$$u_{ml}(\theta)=C_m\frac{\sin^l\theta}{\operatorname{tg}^m\frac{\theta}{2}}. \tag{17}$$

For later computations, it is convenient to introduce the variable $\mu=\cos\theta$. Denote the function $u_{ml}(\arccos\mu)$ by the symbol $P_{ml}(\mu)$. This gives us

$$P_{ml}(\mu)=C_m(1-\mu)^{\frac{l-m}{2}}(1+\mu)^{\frac{l+m}{2}}. \tag{18}$$

In this way, we have defined, except for the constant factor C_m, all of the elements of the right-hand column of the matrix T_g, that is, the functions $T_{ml}(\phi_1, \theta, \phi_2)$. Leaving C_m for the moment undetermined, we find all of the other elements $T_{mn}(\phi_1, \theta, \phi_2)$. To do this, we apply the operator H_- to the already determined functions T_{ml} and make use of the fact that $H_-T_{mn} = a_nT_{m,n-1}$, where $a_n = \sqrt{(l+n)(l-n+1)}$ (see §2, paragraph 3). We substitute the operator H_- from (12) and the functions $T_{mn}(\phi_1, \theta, \phi_2)$ from (15) in this equation. This produces the following relation which serves to define the functions $u_{mn}(\theta)$:

$$\frac{du_{mn}}{d\theta}-\frac{m-n\cos\theta}{\sin\theta}u_{mn}=-i\alpha_nu_{m,n-1}.$$

Upon introducing the variable $\mu=\cos\theta$ and the notation $u_{mn}(\theta) = P_{mn}(\mu)$, this equation becomes

$$(1-\mu^2)^{\frac{1}{2}}\frac{dP_{mn}}{d\mu}+\frac{m-n\mu}{(1-\mu^2)^{\frac{1}{2}}}P_{mn}(\mu)=i\alpha_n P_{m,n-1}(\mu). \tag{19}$$

We set

$$P_{mn}(\mu)=(1-\mu)^{-\frac{n-m}{2}}(1+\mu)^{-\frac{n+m}{2}}v_{mn}(\mu). \tag{20}$$

Substituting this expression for $P_{mn}(\mu)$ in formula (19), we obtain a simple defining relation for $v_{mn}(\mu)$:

$$\frac{dv_{mn}}{d\mu}=i\alpha_n v_{m,\,n-1}(\mu). \tag{21}$$

The functions $P_{ml}(\mu)$ have already been determined in (18). Rewriting these functions in the form (20), we obtain

$$P_{ml}(\mu)=C_m(1-\mu)^{-\frac{l-m}{2}}(1+\mu)^{-\frac{l+m}{2}}v_{ml}(\mu).$$

This shows that

$$v_{ml}(\mu)=C_m(1-\mu)^{l-m}(1+\mu)^{l+m}.$$

From this and from formula (21), we obtain at once that

$$v_{mn}(\mu)=(-i)^{l-n}\frac{C_m}{\alpha_l\alpha_{l-1}\cdots\alpha_{n+1}}\frac{d^{l-n}}{d\mu^{l-n}}[(1-\mu)^{l-m}(1+\mu)^{l+m}].$$

Therefore the functions $P_{mn}(\mu)$, which we shall also designate as $p^l_{mn}(\mu)$, have the form

$$P^l_{mn}(\mu)=$$

$$=(-i)^{l-n}\frac{C_m}{\alpha_l\alpha_{l-1}\cdots\alpha_{n+1}}(1-\mu)^{-\frac{n-m}{2}}(1+\mu)^{-\frac{n+m}{2}}\frac{d^{l-n}}{d\mu^{l-n}}[(1-\mu)^{l-m}(1+\mu)^{l+m}].$$

Substituting now in formula (15) $P^l_{mn}(\cos\theta)$ for $u_{mn}(\theta)$, we obtain the functions $T_{mn}(\phi_1,\theta,\phi_2)$ for all values of the indices m and n.

Nevertheless, there still remain $2l+1$ undertermined constants C_m in the expressions for $T_{mn}(\phi_1,\theta,\phi_2)$. We determine these constants from the condition that the rotation with Euler's angles zero, $g_0=g(0,0,0)$, correspond under the representation to the matrix E. This implies that $T_{mn}(0,0,0)=1$. Hence

$$P^l_{mm}(1)=(-i)^{l-m}\frac{C_m}{\alpha_l\alpha_{l-1}\cdots\alpha_{m+1}}2^{-m}(-1)^{l-m}(l-m)!\;2^{l+m}=1.$$

Consequently, replacing $a_l, a_{l-1},\cdots, a_{m+1}$ by their known values, we find

$$C_m=\frac{(-1)^{l-m}i^{l-m}}{2^l(l-m)!}\sqrt{\frac{(2l)!\,(l-m)!}{(l+m)!}}.$$

Finally, substituting the known values for C_m and $a_l, a_{l-1},\cdots, a_{n+1}$ in the expression for $P^l_{mn}(\mu)$, we obtain

$$P^l_{m,n}(\mu)=\frac{(-1)^{l-m}i^{n-m}}{2^l(l-m)!}\times$$

$$\sqrt{\frac{(l-m)!\,(l+n)!}{(l+m)!\,(l-n)!}}(1-\mu)^{-\frac{n-m}{2}}(1+\mu)^{-\frac{n+m}{2}}\frac{d^{l-n}}{d\mu^{l-n}}[(1-\mu)^{l-m}\cdot(1+\mu)^{l+m}]. \tag{22}$$

We have thus shown that *the matrix corresponding under the irreducible representation with weight l to the rotation g with Euler angles ϕ_1,θ,ϕ_2 is written in* *the canonical basis as follows:*

$$T_g^l = \| T_{mn}^l(\varphi_1, \theta, \varphi_2) \| \qquad (m, n = -l, -l+1, \ldots, l).$$

Here

$$T_{mn}^l(\varphi_1, \theta, \varphi_2) = e^{-im\varphi_2} P_{mn}^l(\cos\theta)\, e^{-in\varphi_1}$$

and

$$P_{mn}^l(\mu) = A(1-\mu)^{-\frac{n-m}{2}}(1+\mu)^{-\frac{n+m}{2}} \frac{d^{l-n}}{d\mu^{l-n}}[(1-\mu)^{l-m}(1+\mu)^{l+m}]. \tag{23}$$

The constant A is equal to $\frac{(-1)^{l-m}\, i^{n-m}}{2^l(l-m)!}\sqrt{\frac{(l-m)!\,(l+n)!}{(l+m)!\,(l-n)!}}$. The functions $T_{mn}^l(\phi_1, \theta, \phi_2)$ will be referred to in the sequel as *generalized spherical functions of order* l.

As examples, we write down the generalized spherical functions of orders ½, *1*, and *2*. Since we know how these functions depend upon ϕ_1 and ϕ_2, we shall save space by writing down the matrices of functions $P_{mn}^{1/2}(\cos\theta)$, $P_{mn}^{1}(\cos\theta)$, and $P_{mn}^{2}(\cos\theta)$. They have the following form:

$$l = \frac{1}{2}$$

$$\begin{Vmatrix} \frac{1}{\sqrt{2}}(1+\cos\theta)^{\frac{1}{2}} & \frac{i}{\sqrt{2}}(1-\cos\theta)^{\frac{1}{2}} \\ \frac{i}{\sqrt{2}}(1-\cos\theta)^{\frac{1}{2}} & \frac{1}{\sqrt{2}}(1+\cos\theta)^{\frac{1}{2}} \end{Vmatrix} = \begin{Vmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{Vmatrix},$$

$$l = 1$$

$$\begin{Vmatrix} \frac{1}{2}(1+\cos\theta) & \frac{i}{\sqrt{2}}\sin\theta & \frac{1}{2}(\cos\theta-1) \\ \frac{i}{\sqrt{2}}\sin\theta & \cos\theta & \frac{i}{\sqrt{2}}\sin\theta \\ \frac{1}{2}(\cos\theta-1) & \frac{i}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta) \end{Vmatrix},$$

$$l = 2$$

$$\begin{Vmatrix} \frac{1}{4}(\cos\theta+1)^2 & \frac{i}{2}\sin\theta(\cos\theta+1) & -\frac{1}{2}\sqrt{\frac{3}{2}}(1-\cos^2\theta) & \frac{i}{2}\sin\theta(\cos\theta-1) & \frac{1}{4}(\cos\theta-1)^2 \\ \frac{i}{2}\sin\theta(\cos\theta+1) & \frac{1}{2}(2\cos^2\theta+\cos\theta-1) & \sqrt{\frac{3}{2}}\, i\sin\theta\cos\theta & \frac{1}{2}(2\cos^2\theta-\cos\theta-1) & \frac{i}{2}\sin\theta(\cos\theta-1) \\ -\frac{1}{2}\sqrt{\frac{3}{2}}(1-\cos^2\theta) & \sqrt{\frac{3}{2}}\, i\sin\theta\cos\theta & \frac{1}{2}(3\cos^2\theta-1) & \sqrt{\frac{3}{2}}\, i\sin\theta\cos\theta & -\frac{1}{2}\sqrt{\frac{3}{2}}(1-\cos^2\theta) \\ \frac{i}{2}\sin\theta(\cos\theta-1) & \frac{1}{2}(2\cos^2\theta-\cos\theta-1) & \sqrt{\frac{3}{2}}\, i\sin\theta\cos\theta & \frac{1}{2}(2\cos^2\theta+\cos\theta-1) & \frac{i}{2}\sin\theta(\cos\theta+1) \\ \frac{1}{4}(\cos\theta-1)^2 & \frac{i}{2}\sin\theta(\cos\theta-1) & -\frac{1}{2}\sqrt{\frac{3}{2}}(1-\cos^2\theta) & \frac{i}{2}\sin\theta(\cos\theta+1) & \frac{1}{4}(\cos\theta+1)^2 \end{Vmatrix}.$$

In these examples, the rows are numbered from top to bottom and the columns from left to right, through the indices $-l, -l+1, \cdots, l$. To obtain the generalized spherical function $T_{mn}^l(\phi_1, \theta, \phi_2)$, it is necessary to multiply the entry in the m-th row and n-th column by the function $e^{-im\phi_2}e^{-in\phi_1}$.

For l an integer and $m=0$, the functions $T_{mn}^l(\phi_1, \theta, \phi_2)$ have the form

$$T^l_{0n}(\varphi_1,\ \theta,\ \varphi_2)=e^{-in\varphi_1}(-1)^l i^n \frac{1}{2^l\cdot l!}\sqrt{\frac{(l+n)!}{(l-n)!}}\,(1-\mu^2)^{-\frac{n}{2}}\frac{d^{l-n}}{d\mu^{l-n}}(1-\mu^2)^l.$$

Comparing this function with the corresponding spherical function $Y_l^n(\phi,\theta)$ of order l, as given in formula (14) of §3, we find that

$$T^l_{0n}(\varphi_1,\ \theta,\ \varphi_2)=\sqrt{\frac{2}{2l+1}}\,Y_l^n\left(\frac{\pi}{2}-\varphi_1,\ \theta\right). \tag{24}$$

This implies that the elements of the zero-th (center) row of the matrix T_g (T_g has a center row, obviously, only when it is a matrix of odd order, and this occurs only when l is an integer) coincide with the spherical functions of order l, up to a constant factor $\sqrt{\frac{2}{2l+1}}$ and after a change of the variable ϕ to $\frac{\pi}{2}-\phi$.* In particular, $P^l_{00}(\mu)$ coincides with the ordinary (that is, not normalized by $\int_{-l}^{l}P^2(\mu)d\mu=l$) Legendre polynomial.

The functions $P^l_{mn}(\mu)$, for arbitrary m and n, are closely connected with the Jacobi polynomials, which are also encountered in analysis. The relevant formula will be given a little later.

We now deduce a number of properties of the functions $P^l_{mn}(\mu)$ from the fact that the matrices T_g are unitary. Suppose that the rotation g is given by its Euler angles ϕ_1,θ,ϕ_2. Then the rotation g^{-1} has Euler angles $\pi-\phi_2,\ \theta,\ \pi-\phi_1$ (see §1). Since T_g is unitary, we have

$$T_g^*=T_g^{-1}=T_{g^{-1}}.$$

All of this produces the relation

$$\overline{T_{nm}(\varphi_1,\ \theta,\ \varphi_2)}=T_{mn}(\pi-\varphi_2,\ \theta,\ \pi-\varphi_1).$$

Cancelling the factor $e^{im\phi_2}e^{in\phi_1}$, we obtain

$$\overline{u_{nm}(\theta)}=u_{mn}(\theta)\cdot(-1)^{m+n}. \tag{25'}$$

Then, setting $\cos\theta\ \mu$, we obtain

$$P^l_{mn}(\mu)=\overline{P^l_{nm}(\mu)}\cdot(-1)^{m+n}. \tag{25}$$

In view of (23), $\overline{P^l_{nm}(\mu)}$ contains the factor i^{n-m}. Hence $\overline{P^l_{nm}(\mu)}=(-1)^{n-m}P^l_{nm}(\mu)$. Thus, we finally obtain

$$P^l_{mn}(\mu)=P^l_{nm}(\mu). \tag{26}$$

That is, *the matrix of functions $P^l_{mn}(\mu)$ is symmetric about its main diagonal.*

It follows from this that the function $P^l_{mn}(\mu)$, in accordance with formulas (23), can be written in the following form:

$$\left.\begin{aligned}&P^l_{mn}(\mu)=A'(1-\mu)^{-\frac{m-n}{2}}(1+\mu)^{-\frac{m+n}{2}}\frac{d^{l-m}}{d\mu^{l-m}}\left[(1-\mu)^{l-n}(1+\mu)^{l+n}\right],\\ &\text{where}\\ &A'=(-1)^{l-n}\frac{i^{m-n}}{2^l(l-n)!}\sqrt{\frac{(l-n)!\,(l+m)!}{(l+n)!\,(l-m)!}}.\end{aligned}\right\} \tag{23'}$$

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* Formula (24) can be derived by using the fact that the elements of the zero-th row does not depend upon ϕ_2 and may be considered as functions on the surface of a sphere (see §3).

Yet another symmetry property of the functions $P^l_{mn}(\mu)$ follows from the following observation. Let us consider the rotation g_0 with Euler angles $(0, \pi, 0)$ (this is a rotation through 180° around the axis ox). For this rotation, we have $\mu = \cos\theta = -1$, so that $\mu + 1 = 0$, and the elements of the matrix T_{g_0} corresponding to this rotation are zero for $m + n \neq 0$ (see formula (23)). For $n = -m$, we have

$$T_{m,-m} = P_{m,-m}(-1) = (-1)^l.$$

Now, let g be an arbitrary rotation with Euler angles ϕ_1, θ, ϕ_2, and let $T_{mn}(\phi_1, \theta, \phi_2)$ be an element of the matrix T_g which corresponds to this rotation. Consider the rotation $\tilde{g} = g_0 g g_0^{-1}$. The matrix corresponding to $\tilde{g}$ is $T_{\tilde{g}} = T_{g_0} T_g T_{g_0^{-1}}$. Multiplying the corresponding matrices, it is easy to see that the element in the m-th row and n-column of the matrix $T_{\tilde{g}}$ is equal to $T_{-m,-n}(\phi_1, \theta, \phi_2)$. On the other hand, the rotation changes the direction of the two co-ordinate axes oy and oz. Hence, if $g = g(\phi_1, \theta, \phi_2)$, it follows that $\tilde{g} = g_0 g g_0^{-1} = g(-\phi_1, \theta, -\phi_2)$.* Consequently, $T_{\tilde{g}} = T(-\phi_1, \theta, -\phi_2)$, and we have the equality

$$T_{-m,-n}(\varphi_1, \theta, \varphi_2) = T_{mn}(-\varphi_1, \theta, -\varphi_2).$$

Cancelling the factor $e^{im\phi_2} e^{in\phi_1}$ and setting $\cos\theta = \mu$, we thus obtain

$$P^l_{-m,-n}(\mu) = P^l_{mn}(\mu). \tag{27}$$

This symmetry property of the functions $P^l_{mn}(\mu)$ shows immediately that these functions depend actually not on the indices m and n, but on $|m+n|$ and $|m-n|$.

We point out, finally, the connection between the functions $P^l_{mn}(\mu)$ and the Jacobi polynomials, referred to above. Jacobi polynomials are polynomials of the form

$$P_s^{\alpha\beta}(\mu) = \frac{(-1)^s}{2^s \cdot s!}(1-\mu)^{-\alpha}(1+\mu)^{-\beta}\frac{d^s}{d\mu^s}\left[(1-\mu)^{s+\alpha}(1+\mu)^{s+\beta}\right].$$

It is evident that if we set $s = l - \frac{1}{2}(|m+n| + |m-n|)$, $\alpha = |n-m|$, and $\beta = |n+m|$, then the functions $P^l_{mn}(\mu)$ can be expressed in terms of the Jacobi polynomials $P_s^{\alpha\beta}(\mu)$ by means of the formula

$$P^l_{mn}(\mu) = K(1-\mu)^{\frac{\alpha}{2}}(1+\mu)^{\frac{\beta}{2}} P_s^{\alpha\beta}(\mu),$$

where K is a constant.

We also remark that since the matrices $T^l_{mn}(g)$ are unitary, we have the identity

$$\sum_{n=-l}^{n=+l} |P^l_{mn}(\cos\theta)|^2 \equiv 1.$$

5. The addition formula for matrix elements. Equality (3) of the present section, namely, $T_{mn}(g'g'') = \sum_{s=-l}^{s=l} T_{ms}(g')T_{sn}(g'')$, is essentially an addition formula for generalized spherical functions. As a special case, it contains the usual addition formula for Legendre polynomials. We shall now give an explicit statement of the general addition formula. Suppose that the rotation g' is given by the Euler angles ϕ_1', θ', 0, and that the rotation g'' is given by the Euler angles 0, θ'', ϕ_2''. Let the rotation $g = g'g''$ be given by the Euler angles ϕ_1, θ, ϕ_2.* Then

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* As a matter of fact, the matrix $\tilde{g} = g_0 g g_0^{-1}$ can be considered as the matrix of the same rotation g, but expressed in a new system of co-ordinates, obtained from the original system by the rotation g_0. But this system differs from the original only in that the directions of the axes oy and oz are changed into their negatives.

$$T_{mn}(\varphi_1, \theta, \varphi_2) = \sum_{s=-l}^{s=l} T_{ms}(\varphi_1', \theta', 0)\, T_{sn}(0, \theta'', \varphi_2'').$$

Replacing $T_{mn}(\phi_1, \theta, \phi_2)$ by $e^{-im\phi_2} P^l_{mn}(\cos\theta) e^{-in\phi_1}$ and writing the sum $\phi_1' + \phi_2''$ as ϕ, we obtain the equality

$$e^{-im\varphi_2} P^l_{mn}(\cos\theta)\, e^{-in\varphi_1} = \sum_{s=-l}^{s=l} e^{-is\varphi} P^l_{ms}(\cos\theta')\, P^l_{sn}(\cos\theta''). \tag{28}$$

It remains to express ϕ_1, θ, and ϕ_2 in terms of ϕ, θ' and θ''. For this purpose, we consider the matrices $T_{g'}$, $T_{g''}$, and T_g for $l = ½$. The equality $T_{g'}T_{g''} = T_g$ may be written as

$$\left\| \begin{matrix} \cos\frac{\theta'}{2} e^{i\frac{\varphi_1'}{2}} & -i\sin\frac{\theta'}{2} e^{-i\frac{\varphi_1'}{2}} \\ -i\sin\frac{\theta'}{2} e^{i\frac{\varphi_1'}{2}} & \cos\frac{\theta'}{2} e^{-i\frac{\varphi_1'}{2}} \end{matrix} \right\| \left\| \begin{matrix} \cos\frac{\theta''}{2} e^{i\frac{\varphi_2''}{2}} & -i\sin\frac{\theta''}{2} e^{i\frac{\varphi_2''}{2}} \\ -i\sin\frac{\theta''}{2} e^{-i\frac{\varphi_2''}{2}} & \cos\frac{\theta''}{2} e^{-i\frac{\varphi_2''}{2}} \end{matrix} \right\| =$$

$$= \left\| \begin{matrix} \cos\frac{\theta}{2} e^{i\frac{\varphi_1+\varphi_2}{2}} & -i\sin\frac{\theta}{2} e^{i\frac{\varphi_2-\varphi_1}{2}} \\ -i\sin\frac{\theta}{2} e^{-i\frac{\varphi_2-\varphi_1}{2}} & \cos\frac{\theta}{2} e^{-i\frac{\varphi_1+\varphi_2}{2}} \end{matrix} \right\|.$$

This gives us two complex equations for defining θ', θ'', and $\phi = \phi_1' + \phi_2''$:

$$\cos\frac{\theta}{2} e^{i\frac{\varphi_1+\varphi_2}{2}} = \cos\frac{\theta'}{2}\cos\frac{\theta''}{2} e^{i\frac{\varphi}{2}} - \sin\frac{\theta'}{2}\sin\frac{\theta''}{2} e^{-i\frac{\varphi}{2}},$$

$$\sin\frac{\theta}{2} e^{i\frac{\varphi_2-\varphi_1}{2}} = \cos\frac{\theta'}{2}\sin\frac{\theta''}{2} e^{i\frac{\varphi}{2}} + \sin\frac{\theta'}{2}\cos\frac{\theta''}{2} e^{-i\frac{\varphi}{2}}.$$

We equate the absolute value and argument of the left and right sides of these equations, and obtain the formulas

$$\left.\begin{aligned} \cos\theta &= \cos\theta'\cos\theta'' - \sin\theta'\sin\theta''\cos\varphi, \\ \operatorname{tg}\varphi_1 &= \frac{\sin\varphi\sin\theta''}{\cos\theta'\sin\theta''\cos\varphi + \cos\theta''\sin\theta'}, \\ \operatorname{tg}\varphi_2 &= \frac{\sin\varphi\sin\theta'}{\sin\theta'\cos\theta''\cos\varphi + \cos\theta'\sin\theta''}. \end{aligned}\right\} \tag{29}$$

In this fashion, we arrive finally at the following result. *Let ϕ, θ', and θ'' be arbitrary angles such that $0 \le \phi < 2\pi$, $0 \le \theta'$, $\theta'' < \pi$, and let θ, ϕ_1, and ϕ_2 be defined from ϕ, θ', and θ'' by means of the formulas* (29). *Then the following addition formula holds for generalized spherical functions:*

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* We actually lose no generality in supposing that $\phi_2' = \phi_1'' = 0$. It is obvious that if we add the angle α to ϕ_1'', then we add the same angle α to ϕ_1. Adding the angle α to ϕ_2', we change ϕ_2 in the same way. Therefore, for arbitrary ϕ_2' and ϕ_1'', we have merely cancelled the factor $e^{-im\phi_2'} e^{-in\phi_1''}$ from the corresponding formula.

$$e^{-im\;{}_2} P^l_{mn}(\cos\theta)\, e^{-in\varphi_1} = \sum_{s=-l}^{s=l} e^{-is\varphi} P^l_{ms}(\cos\theta')\, P^l_{sn}(\cos\theta''). \qquad (28')$$

For the special case $m = n = 0$, formula (28') becomes the usual addition formula for Legendre polynomials, which expresses the Legendre polynomial of $\cos\theta = \cos\theta' \cos\theta'' - \sin\theta' \sin\theta'' \cos\phi$ in terms of the associated functions of θ' and θ''.

Upon setting $m = 0$, we obtain the addition theorem for ordinary spherical functions $P^l_{0n}(\cos\theta)e^{-in\phi_1} = Y^n_l((\pi/2)-\phi_1, \theta)$:

$$P^l_{0n}(\cos\theta) = \sum_{s=-l}^{s=l} e^{-is\varphi} P^l_{0s}(\cos\theta')\, P^l_{sn}(\cos\theta'').$$

The addition formula assumes an especially simple form for $\phi = 0$ (in this case, $\phi_1 = \phi_2 = 0$). In this case, we also have $\theta = \theta' + \theta''$, and the addition formula becomes

$$P^l_{mn}[\cos(\theta' + \theta'')] = \sum_{s=-l}^{s=l} P^l_{ms}(\cos\theta')\, P^l_{sn}(\cos\theta''). \qquad (30)$$

If we recall the connection between the functions $P^l_{mn}(\mu)$ and the Jacobi polynomials (see paragraph 4), we can interpret this identity as an addition formula for Jacobi polynomials. Since

$$P^l_{0n}[\cos(\theta' + \theta'')] = \sum_{s=-l}^{l} P^l_{0s}(\cos\theta')\, P^l_{sn}(\cos\theta''),$$

and since $P^l_{0n}[\cos(\theta' + \theta'')]$ are the associated Legendre functions, we see that the generalized spherical functions arise in a natural way upon expanding the functions $P^l_{0n}[\cos(\theta' + \theta'')]$ in series of associated functions of θ'. It follows from formulas (28') and (30) that these same generalized spherical functions form a system which is closed under application of the addition formula. The addition formula also makes it possible to express a Legendre polynomial of the cosine of the sum of several angles by means of functions each of which depends upon only one of these angles. Thus, for example,

$$P_l[\cos(\theta' + \theta'' + \theta''')] = \sum_{s_1=-l}^{l} \sum_{s_2=-l}^{l} P^l_{0s_1}(\cos\theta')\, P^l_{s_1s_2}(\cos\theta'')\, P^l_{s_20}(\cos\theta''').$$

6. Expansion of functions on the rotation group in series of generalized spherical functions. In order to identify the matrix elements $T_{mn}(g)$, we defined at the beginning of the present section certain irreducible representations of the rotation group by means of transformations U_{g_1}. It will be recalled that these transformations are defined on finite-dimensional spaces R^m of functions defined on the rotation group.

We consider now the set of all functions of g,

$$f(g) = f(\phi_1, \theta, \phi_2),$$

for which the integral

$$\int |f(g)|^2\,dg = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} |f(\varphi_1, \theta, \varphi_2)|^2 \sin\theta\, d\varphi_1\, d\theta\, d\varphi_2 \tag{31}$$

converges. We define the scalar product of $f_1(g)$ and $f_2(g)$ by the formula

$$(f_1, f_2) = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f_1(\varphi_1, \theta, \varphi_2)\,\overline{f_2(\varphi_1, \theta, \varphi_2)}\sin\theta\, d\varphi_1\, d\theta\, d\varphi_2. \tag{32}$$

With this scalar product, the space of functions forms a Hilbert space of functions on the group. The transformations

$$U_{g_1} f(g) = f(gg_1)$$

comprise an infinite-dimensional representation in this Hilbert space of the rotation group.* It is called *the regular representation* of the rotation group. We know the irreducible representations into which it is decomposable. They are, in fact, the representations defined in the subspaces of generalized spherical functions $T^l_{mn}(\phi_1, \theta, \phi_2)$, where l and m are fixed. It is not difficult to convince one's self that these are the only irreducible components of the regular represéntaion. To see this, we argue as follows. In every subspace of the space of square-integrable functions in which there operates an irreducible representation of weight l, there must, in accordance with the general theory, exist an element f satisfying the equations $H_3 f = lf$, $H_+ f = 0$. From formula (12) of the present section, we see that f must satisfy the differential equations

$$i\frac{\partial f}{\partial \phi_1} = lf \text{ and } \operatorname{ctg}\theta\frac{\partial f}{\partial \phi_1} - \frac{1}{\sin\theta}\,\frac{\partial f}{\partial \phi_2} + i\frac{\partial f}{\partial \phi} = 0.$$

Any solution of the first equation has the form $f = F(\theta, \phi_2)e^{-il\phi_1}$, and the second, after substitution of this last function, becomes

$$\frac{\partial F}{\partial \theta} + \frac{i}{\sin\theta}\,\frac{\partial F}{\partial \phi_2} - l\operatorname{ctg}\theta F = 0.$$

The solutions of this equation which are periodic with respect to ϕ_2 can be written in the form

$$F(\theta, \varphi_2) = \sum_m u_m(\theta)e^{-im\varphi_2},$$

where $u_m(\theta)$ satisfies the equation

$$\frac{du}{d\theta} + \frac{m - l\cos\theta}{\sin\theta}u = 0.$$

It follows from this (see also (17) of the present section) that

$$u_m(\theta) = C\frac{\sin^l\theta}{\operatorname{tg}^m\frac{\theta}{2}}.$$

The possible values of m are determined from the condition that the functions $u_m(\theta)e^{-im\phi_2}e^{-il\phi_1}$ should belong to the Hilbert space under consideration, that is, that the integral

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* This representation is obviously unitary, because the scalar product of the functions $f_1(g)$ and $f_2(g)$ is invariant under multiplication on the right by the element g_0 (see §1, paragraph 3).

$$\int_0^\pi |u_m(\theta)|^2 \sin\theta\, d\theta$$

should converge. When we set $\cos\theta = \mu$, it is easy to see that this integral converges just for those values of m such that $-l \leq m \leq l$. But in this case, the solutions we have found are linear combinations of generalized spherical functions.

Therefore, the decomposition of the regular representation into irreducible components implies that every single-valued function on the rotation group with integrable square modulus can be expanded in a series of generalized spherical functions $T^l_{mn}(\phi_1, \theta, \phi_2)$. The functions $T^l_{mn}(\phi_1, \theta, \phi_2)$ and $T^{l'}_{mn}(\phi_1, \theta, \phi_2)$ are orthogonal functions for $l \neq l'$, since they belong to orthogonal invariant subspaces, in which different irreducible representations of the rotation group operate. For $l' = l$ and $m' \neq m$, they are also orthogonal, since $\int_0^{2\pi} e^{-im\phi_2} e^{im'\phi_2}\, d\phi_2 = 0$. In the same way, we see that the functions are orthogonal for $l' = l$ and $n' \neq n$. The integral of the square of the modulus of each of the functions $T^l_{mn}(\phi_1, \theta, \phi_2)$ over the group, that is, the integral

$$\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} |T^l_{mn}(\varphi_1, \theta, \varphi_2)|^2 \sin\theta\, d\varphi_1\, d\theta\, d\varphi_2, \text{ is equal to } \frac{2}{2l+1}.$$

Finally, we state the following theorem. *The set of all generalized spherical functions* $T^l_{mn}(\phi_1, \theta, \phi_2)$ *(with integral values of* l*) forms a complete orthogonal system in the space of functions* $f(\phi_1, \theta, \phi_2)$ $(0 \leq \theta \leq \pi,\ 0 \leq \phi_1, \phi_2 < 2\pi)$, *where the scalar product is defined by formula* (32).

Appendix to §7.

Recurrence relations among generalized spherical functions. There exist a whole series of recurrence relations among the generalized spherical functions T^l_{mn}. A part of these connect spherical functions of the same order (that is, with one and the same value of l). Another part of these recurrence relations connect functions of different orders.

We have already encountered recurrence relations connecting spherical functions of a given order (see §7, paragraph 4). They are derived from the relations $H_- T^l_{mn} = a_n T^l_{m,n-1}$ and $H_+ T^l_{mn} = a_{n+1} T^l_{m,n+1}$. After writing the operators H_- and H_+ in terms of formula (12), §7, and replacing T^l_{mn} by $e^{-im\phi_2} u_{mn}(\theta) e^{-in\phi_1}$, these relations can be written in the form *

$$\left.\begin{aligned} \frac{du_{mn}}{d\theta} - \frac{m - n\cos\theta}{\sin\theta} u_{m,n} &= -i\alpha_n u_{m,n-1}, \\ \frac{du_{mn}}{d\theta} + \frac{m - n\cos\theta}{\sin\theta} u_{m,n} &= -i\alpha_{n+1} u_{m,n+1}. \end{aligned}\right\} \qquad (1)$$

When we write $\cos\theta = \mu$, these formulas become

$$\left.\begin{aligned} \sqrt{1-\mu^2}\,\frac{dP^l_{mn}(\mu)}{d\mu} + \frac{m - n\mu}{\sqrt{1-\mu^2}} P^l_{mn}(\mu) &= i\alpha_n P^l_{m,n-1}, \\ \sqrt{1-\mu^2}\,\frac{dP^l_{mn}(\mu)}{d\mu} - \frac{m - n\mu}{\sqrt{1-\mu^2}} P^l_{mn}(\mu) &= i\alpha_{n+1} P^l_{m,n+1}, \end{aligned}\right\} \qquad (1')$$

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* For the sake of brevity, we denote $P^l_{mn}(\cos\theta)$ by $u_{mn}(\theta)$.

where
$$\alpha_n = \sqrt{(l+n)\ (l-n+1)}.$$

It was shown in §7, paragraph 4, formula (26), that $u_{mn}(\theta) = u_{nm}(\theta)$. Replacing $u_{mn}(\theta)$ by $u_{nm}(\theta)$ in (1) and then writing m for n and n for m, we obtain the formulas

$$\left.\begin{aligned} \frac{du_{mn}}{d\theta} - \frac{n-m\cos\theta}{\sin\theta}\,u_{mn} &= -i\alpha_m u_{m-1,\,n}, \\ \frac{du_{mn}}{d\theta} + \frac{n-m\cos\theta}{\sin\theta}\,u_{mn} &= -i\alpha_{m+1} u_{m+1,\,n}. \end{aligned}\right\} \tag{2}$$

From each of the formulas (1) and (2), it is possible to deduce relations between 3 successive elements of a row (or column), not involving derivatives. They are

$$\alpha_{n+1}u_{m,\,n+1} - \alpha_n u_{m,\,n-1} = 2i\,\frac{m-n\cos\theta}{\sin\theta}\,u_{mn}, \tag{3}$$

$$\alpha_{m+1}u_{m+1,\,n} - \alpha_m u_{m-1,\,n} = 2i\,\frac{n-m\cos\theta}{\sin\theta}\,u_{mn}, \tag{3'}$$

where again $u_{mn}(\theta)$ is written instead of $P^l_{mn}(\cos\theta)$.

Formulas of a different sort can be obtained by using a certain property of the matrices of the representation, discussed in §2. In paragraph of §2, we showed that if $\eta = (\eta_1, \eta_2, \eta_3)$ is a certain vector and $A_\eta = A_1\eta_1 + A_2\eta_2 + A_3\eta_3$, where the A_k are the matrices corresponding to infinitesimal rotations around the co-ordinate axes, then, for every rotation g, the equality

$$T_g A_\eta T_{g^{-1}} = A_{\tilde{\eta}}, \tag{4}$$

obtains. Here $\tilde{\eta} = g\eta$ and $A_{\tilde{\eta}} = A_1\tilde{\eta}_1 + A_2\tilde{\eta}_2 + A_3\tilde{\eta}_3$. Multiply the equality (4) on the right by T_g and set $\eta = (1, 0, 0)$. Then the components of the vector $\tilde{\eta}$ will be elements of the first column of the matrix $\|g_{ik}\|$. In §1, we wrote the matrix $\|g_{ik}\|$ as a function of the Euler angles of g (see formula (9) of §1). Substituting the elements g_{11}, g_{21}, g_{31} of this matrix in the equality $A_1T_g = g_{11}T_gA_1 + g_{21}T_gA_2 + g_{31}T_gA_3$, we obtain the equality

$$\begin{aligned} T_gA_1 = (\cos\varphi_2\cos\varphi_1 - \cos\theta\sin\varphi_2\sin\varphi_1)\,A_1T_g + \\ + (\sin\varphi_2\cos\varphi_1 + \cos\theta\cos\varphi_2\sin\varphi_1)\,A_2T_g + \sin\varphi_1\sin\theta\,A_3T_g. \end{aligned}$$

For further calculations, it is convenient to rewrite the matrices A_k in terms of the matrices H_+, H_-, and H_3, in accordance with the formulas $A_1 = -\frac{i}{2}(H_+ + H_-)$, $A_2 = \frac{1}{2}(H_- - H_+)$, and $A_3 = -iH_3$. Doing this and collecting the coefficients of H_+ and H_- on the right side, we find

$$\begin{aligned} T_g(H_+ + H_-) = (\cos\varphi_1 - i\cos\theta\sin\varphi_1)\,e^{-i\varphi_2}H_+T_g + \\ + (\cos\varphi_1 + i\cos\theta\sin\varphi_1) + e^{i\varphi_2}H_-T_g - 2\sin\varphi_1\sin\theta\,H_3T_g. \end{aligned}$$

We now apply the transformations standing on the two sides of the last equality to a vector f_n of the canonical basis and equate the coefficients of f_m in the expressions so obtained. We make use of the facts that $T_gf_n = \Sigma_m T_{mn}f_m$, $H_+f_n = \alpha_{n+1}f_{n+1}$, $H_-f_n = \alpha_nf_{n-1}$, and $H_3f_n = nf_n$, Doing all this, we find the following relation among the functions T_{mn}:

$$\begin{aligned} \alpha_{n+1}T_{m,\,n+1} + \alpha_nT_{m,\,n-1} = (\cos\varphi_1 - i\cos\theta\sin\varphi_1)\,e^{-i\varphi_2}\alpha_mT_{m-1,\,n} + \\ + (\cos\varphi_1 + i\cos\theta\sin\varphi_1)\,e^{i\varphi_2}\alpha_{m+1}T_{m+1,\,n} - 2m\sin\varphi_1\sin\theta T_{m,n}. \end{aligned}$$

In this relation, we replace $T_{m,n}$ by $e^{-im\phi_2}u_{mn}(\theta)e^{-in\phi_1}$ and multiply the entire relation by $e^{im\phi_2+in\phi_1}$. We then obtain a connection among the functions $u_{mn}(\theta)$, valid for all values of ϕ_1:

$$\alpha_{n+1}u_{m,n+1}e^{-i\varphi_1}+\alpha_n u_{m,n-1}e^{i\varphi_1}=$$
$$=(\cos\varphi_1-i\cos\theta\sin\varphi_1)\alpha_m u_{m-1,n}+$$
$$+(\cos\varphi_1+i\cos\theta\sin\varphi_1)\alpha_{m+1}u_{m+1,n}-2m\sin\varphi_1\sin\theta\, u_{mn}.$$

In this relation, we write $\cos\phi_1=\frac{e^{i\phi_1}+e^{-i\phi_1}}{2}$ and $\sin\phi_1=\frac{e^{i\phi_1}-e^{-i\phi_1}}{2i}$. Equating the coefficients of $e^{i\phi_1}$ and also of $e^{-i\phi_1}$ on the left and right sides, we obtain the final formulas

$$\alpha_{n+1}u_{m,n+1}=\frac{1}{2}(1+\cos\theta)\alpha_m u_{m-1,n}+\frac{1}{2}(1-\cos\theta)\,\alpha_{m+1}u_{m+1,n}-im\sin\theta\, u_{mn}, \tag{5}$$

$$\alpha_n u_{m,n-1}=\frac{1}{2}(1-\cos\theta)\alpha_m u_{m-1,n}+\frac{1}{2}(1+\cos\theta)\,\alpha_{m+1}u_{m+1,n}+im\sin\theta u_{mn}. \tag{5'}$$

These formulas connect 3 adjacent elements of the n-th column with an element of the $(n+1)$st or $(n-1)$st column. Here, as usual, we denote $P^l_{mn}(\cos\theta)$ by $u_{mn}(\theta)$. Using the fact that $u_{mn}(\theta)=u_{nm}(\theta)$, we can obtain, just as was done above, analogous formulas which connect 3 adjacent elements of a row with an element of the next row above or below:

$$\alpha_{m+1}u_{m+1,n}=\frac{1}{2}(1+\cos\theta)\,\alpha_n u_{m,n-1}+\frac{1}{2}(1-\cos\theta)\,\alpha_{n+1}u_{m,n+1}-in\sin\theta\, u_{mn}, \tag{6}$$

$$\alpha_m u_{m-1,n}=\frac{1}{2}(1-\cos\theta)\,\alpha_n u_{m,n-1}+\frac{1}{2}(1+\cos\theta)\,\alpha_{n+1}u_{m,n+1}+in\sin\theta\, u_{mn}. \tag{6'}$$

If we take $\eta=(0,1,0)$ in (4), we get just the same formulas (5), (5′). However, if we take $\eta=(0,0,1)$, then we obtain the formula

$$\alpha_{m+1}u_{m+1,n}-\alpha_m u_{m-1,n}=2i\frac{n-m\cos\theta}{\sin\theta}u_{mn}.$$

We turn now to the derivation of formulas which relate elements T^l_{mn} of the matrices T_g with different values of l (thus corresponding to distinct irreducible representations). In order to do this, we avail ourselves of the results of paragraph 4, §4, concerning the decomposition into irreducible parts of the product of two irreducible representations. We saw there that the product $g\to T_g$ of the irreducible representation of weight l with the irreducible representation of weight l can be decomposed into 3 irreducible parts: $g\to T_g^{l+1}$ of weight $l+1$, $g\to T_g^l$ of weight l and $g\to T_g^{l-1}$ of weight $l-1$. The space $R_1\times R_2$, in which the product representation operates, has a natural basis e_kf_{m-k}. The subspaces in which the 3 irreducible parts operate have canonical bases $\{g_m^{l+1}\}$, $\{g_m^l\}$, and $\{g_m^{l-1}\}$. The connection among these bases is expressed as follows:

$$\left.\begin{aligned} e_{-1}f_{m+1} &= c_{11}^{m}g_{m}^{l+1} + c_{12}^{m}g_{m}^{l} + c_{13}^{m}g_{m}^{l-1}, \\ e_{0}f_{m} &= c_{21}^{m}g_{m}^{l+1} + c_{22}^{m}g_{m}^{l} + c_{23}^{m}g_{m}^{l-1}, \\ e_{1}f_{m-1} &= c_{31}^{m}g_{m}^{l+1} + c_{32}^{m}g_{m}^{l} + c_{33}^{m}g_{m}^{l-1}. \end{aligned}\right\} \qquad (7)$$

The values of the coefficients c_{ik}^{m} are given in formula (9) of §4.

We now write T_g^l for the matrix of the irreducible representation of weight l in the canonical basis. Apply the transformation T_g to both sides of each of the equalities (7). On the left sides, using the definition of the product of representations, we obtain

$$T_g e_k f_{m-k} = T_g^1 e_k T_g^l f_{m-k} = c_{k+2,1}^{m} T_g^{l+1} g_m^{l+1} + c_{k+2,2}^{m} T_g^{l} g_m^{l} + c_{k+2,3}^{m} T_g^{l-1} g_m^{l-1},$$

$$(k = -1,\ 0,\ 1).$$

Next, write the elements of the matrices T_g^l as T_{mn}^l. The last equality gives us the relation

$$(T_{-1k}^{1} e_{-1} + T_{0,k}^{1} e_0 + T_{1k}^{1} e_1) \sum T_{j,\,m-k}^{l} f_j =$$

$$= \sum c_{k+2}^{m} T_{jm}^{l+1} g_j^{l+1} + c_{k+2,2}^{m} T_{j,m}^{l} g_j^{l} + c_{k+2,3}^{m} T_{jm}^{l-1} g_j^{l-1}.$$

Now, in the right side, rewrite the vectors g_j^{l+1}, g_j^l, g_j^{l-1} in terms of $e_{-1}f_{j+1}$, e_0f_j, e_1f_{j-1} by use of the formulas (7).* We then equate the coefficients of the vectors $e_{-1}f_{j+1}$, e_0f_j, and e_1f_{j-1} on the left and right sides of the equality so obtained. We obtain 3 relations, which depend upon k. In each of these relations, give the number k its 3 possible values -1, 0, 1, and substitute the functions $T_{mn}^l(\phi_1, \theta, \phi_2)$ (see the second matrix in the table on page 278). We obtain 9 recurrence relations:

$$\left.\begin{aligned} c_{11}^{m} T_{jm}^{l+1} c_{11}^{j} + c_{12}^{m} T_{jm}^{l} c_{12}^{j} + c_{13}^{m} T_{jm}^{l-1} c_{13}^{j} &= \frac{1}{2}(1+\cos\theta)\, e^{i\varphi_2 + i\varphi_1} T_{j+1,\,m+1}^{l}. \\ c_{11}^{m} T_{jm}^{l+1} c_{21}^{j} + c_{12}^{m} T_{jm}^{l} c_{22}^{j} + c_{13}^{m} T_{jm}^{l-} c_{23}^{j} &= \frac{i}{\sqrt{2}} \sin\theta\, e^{i\varphi_2} T_{j,\,m+1}^{l}, \\ c_{11}^{m} T_{jm}^{l+1} c_{31}^{j} + c_{12}^{m} T_{jm}^{l} c_{32}^{j} + c_{13}^{m} T_{jm}^{l-1} c_{33}^{j} &= \frac{1}{2}(\cos\theta - 1)\, e^{i\varphi_2 - i\varphi_1} T_{j-1,\,m+1}^{l}, \\ c_{21}^{m} T_{jm}^{l+1} c_{11}^{j} + c_{22}^{m} T_{jm}^{l} c_{12}^{j} + c_{23}^{m} T_{jm}^{l-1} c_{13}^{j} &= \frac{i}{\sqrt{2}} \sin\theta\, e^{i\varphi_1} T_{j+1,\,m}^{l}. \\ c_{21}^{m} T_{jm}^{l+1} c_{21}^{j} + c_{22}^{m} T_{jm}^{l} c_{22}^{j} + c_{23}^{m} T_{jm}^{l-1} c_{23}^{j} &= \cos\theta\, T_{j,\,m}^{l}, \\ c_{21}^{m} T_{jm}^{l+1} c_{31}^{j} + c_{22}^{m} T_{jm}^{l} c_{32}^{j} + c_{23}^{m} T_{jm}^{l-1} c_{33}^{j} &= \frac{i}{\sqrt{2}} \sin\theta\, e^{-i\varphi_1} T_{j-1,\,m}^{l}, \\ c_{31}^{m} T_{jm}^{l+1} c_{11}^{j} + c_{32}^{m} T_{jm}^{l} c_{12}^{j} + c_{33}^{m} T_{jm}^{l-1} c_{13}^{j} &= \frac{1}{2}(\cos\theta - 1)\, e^{-i\varphi_2 + i\varphi_1} T_{j+1,\,m-1}^{l}, \\ c_{31}^{m} T_{jm}^{l+1} c_{21}^{j} + c_{32}^{m} T_{jm}^{l} c_{22}^{j} + c_{33}^{m} T_{jm}^{l-1} c_{23}^{j} &= \frac{i}{\sqrt{2}} \sin\theta\, e^{-i\varphi_2} T_{j,\,m-1}^{l}, \\ c_{31}^{m} T_{jm}^{l+1} c_{31}^{j} + c_{32}^{m} T_{jm}^{l} c_{32}^{j} + c_{33}^{m} T_{jm}^{l-1} c_{33}^{j} &= \frac{1}{2}(1+\cos\theta)\, e^{-i\varphi_2 - i\varphi_1} T_{j-1,\,m-1}^{l}. \end{aligned}\right\} \qquad (8)$$

The numbers c_{ik}^m are known numbers. They are the elements of the matrix

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* We recall that the matrix $||c_{ik}^m||$ is orthogonal, and hence its inverse is identical with its transpose.

$C^{(m)}$, which is given by formula (9) of §4. We shall not substitute these values of c^m_{ik} in all of the formulas (8), but shall content ourselves with writing out the fifth of these formulas. This gives the connection existing among the functions T^{l+1}_{mn}, T^l_{mn}, and T^{l-1}_{mn}, which are entries at one and the same place in the corresponding matrices. We obtain the following equality:

$$\frac{\sqrt{(l+m+1)(l-m+1)(l+j+1)(l-j+1)}}{(2l+1)(l+1)}T^{l+1}_{jm}+\frac{mj}{l(l+1)}T^l_{jm}+$$

$$+\frac{\sqrt{(l+m)(l-m)(l+j)(l-j)}}{l(2l+1)}T^{l-1}_{jm}=\cos\theta T^l_{jm}. \qquad (9)$$

After multiplying by $e^{-im\phi_2-ij\phi_1}$, we obtain the equality

$$\frac{\sqrt{(l+m+1)(l-m+1)(l+j+1)(l-j+1)}}{(2l+1)(l+1)}P^{l+1}_{jm}(\mu)+\frac{mj}{l(l+1)}P^l_{jm}(\mu)+$$

$$+\frac{\sqrt{(l+m)(l-m)(l+j)(l-j)}}{l(2l+1)}P^{l-1}_{jm}(\mu)=\mu P^l_{jm}(\mu). \qquad (9')$$

For $j=m=0$, this formula becomes the recurrence formula for Legendre polynomials.

§8. Expansion of vector and tensor fields

In §3, we expanded functions of θ and ϕ in series of spherical functions. Such decompositions are usually employed in solving various problems in spherical co-ordinates. In the statement of such problems, there exists spherical symmetry relative to some point—that is, the problems are invariant under rotations.

In the present section, we shall carry out the analogous expansion for functions whose values are not numbers, but vectors, tensors, or any other quantity. Thus the contents of the present section form an extension of the results of §3, where such expansions were given for scalar functions. As in §3, the expansion of functions is obtained by decomposing into irreducible components certain representations of the rotation group which are originally defined as transformations of the functions under study.

Special functions in terms of which the expansion is carried out are the generalized spherical functions, as we shall see further on. These functions, computed in §7, admit the ordinary spherical functions as special cases.

For the sake of clarity, we begin, in paragraph 1, by solving the problem for functions whose values are vectors, and then for functions whose values are arbitrary quantities.

Expansions of this kind for all sorts of quantities are encountered in many physical questions.

The solution of Laplace's equation in spherical co-ordinates leads naturally to the expansion of functions in a series of spherical functions. In just the same way, the solution of Maxwell's equations in spherical co-ordinates leads to series expansions for functions whose values are vectors (see for example, the paper of V. B. Beresteckiĭ* on this subject).

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* V. B. Beresteckiĭ, *Electromagnetic fields of multipoles*, ŽETF, 17, part I, 12–18 (1947).

In the same way, a solution of Dirac's equation in a centrally symmetric field leads to the expansion in series of functions on the sphere whose values are spinors. The functions which are employed for this expansion were introduced by V. A. Fok* and are called spherical functions with spin.

G. I. Petrašen' introduced the concept of spherical vectors in connection with the solution of the equations appearing in the theory of elasticity** and has applied this notion successfully to the solution of problems.

Finally, in a paper by V. B. Beresteckiĭ, A. Z. Dolginov, and K. A. Ter-Martirocyan,***, one finds expansions of functions whose values are arbitrary l-vectors (that is, quantities which are transformed according to the irreducible representation of weight l). The expansion was carried out by means of what are called spherical (l, L)-vector functions. The components of these are linear combinations of ordinary spherical functions. The coefficients of these linear combinations coincide with the coefficients c_{ik}^{m}, defined in paragraph 4 of §4. In the general case, these coefficients are quite difficult to identify.

As remarked above, we shall carry out the expansion of components of quantities by the use of generalized spherical functions. The connection of these functions with ordinary spherical functions can be established by the use of the recurrence relations of §7.

1. Expansion of vector functions. We consider a function $a(x)$, where x is a point of 3-dimensional space and a is a vector. That is, we consider a vector field.

We shall explain how such functions are transformed by rotations. We subject the vector field $a(x)$ to an arbitrary rotation g_0. As a result of this rotation, we obtain a new vector field $a'(x)$. We find the expression of $a'(x)$ in terms of $a(x)$. In the first place, after the rotation g_0 has been carried out, we find at the point x the vector whose initial point was previously at the point $g_0^{-1}x$. In the second place, the vector $a(g_0^{-1}x)$ is not moved to the point x without change, but together with the whole vector field, it is subjected to the rotation g_0.

Consequently, as a result of the rotation g_0, the vector field $a(x)$ goes over into the vector field $a'(x) = g_0 a(g_0^{-1}x)$. Thus, *to every rotation* g_0, *there corresponds a transformation* T_{g_0} *of the vector function* $a(x)$, *defined by the formula*

$$T_{g_0} a(x) = g_0 a(g_0^{-1} x). \tag{1}$$

It is obvious that the transformation T_{g_0} is linear.

Furthermore, it follows from the very definition of the transformation T_{g_0} that the transformation $T_{g_0 g_1}$ which corresponds to the rotation $g_0 g_1$ is the product of the transformations T_{g_0} and T_{g_1}:

$$T_{g_0 g_1} = T_{g_0} T_{g_1}. \tag{2}$$

The points of an arbitrary sphere with center at the origin remain on the same sphere after being subjected to the rotation g_0. For this reason, in studying the transformations T_g, we shall limit ourselves to vector fields defined on the surface of the unit sphere. That is, we shall study vector fields $a(P) = a(\vartheta, \phi)$, where a is a vector depending on the point P of the sphere with spherical co-ordinates ϑ and ϕ. The transformations T_g, as the formula (2) shows, form a representation of the rotation group in the space of vector functions * on the surface of the sphere. We shall determine invariant subspaces of the space of vector functions on the sphere, in which this representation is irreducible. With this end in view, we find certain finite systems of vector functions which go into linear combinations of themselves under the transformations T_{g_0}.

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* V. A. Fok, *Principles of quantum mechanics*, KUBUČ, 1932.

** G. I. Petrašen', *Dynamical problems of the theory of elasticity in the case of an isotrophic sphere.* Učenye zapiski LGU, no. 114, part 17 (1949).

*** V. B. Beresteckiĭ, A. Z. Dolginov, K. A. Ter-Martirocyan, *Angular wave functions of particles with spin*, ŽETF, 20, part 6, 527–537 (1950).

For a field of scalar functions, we have such systems in abundance: the spherical functions of a given order l.

In the case of a vector field, it would also be possible to expand each component of the vector $a(P)$ in a series of spherical functions. However, such an expansion is not convenient, since under rotation, every component of a vector goes into a linear combination of all 3 components. That is, the components get all mixed up.

As a matter of fact, there exists a more convenient method of describing a vector field. For example, for one component of a vector, it is logical to take the component $a_r(P)$ normal to the surface of the sphere. Now, under rotations, the normal component $a_r(P)$ at the point P changes into the normal component of the vector at the point $g_0^{-1}P$. That is,

$$T_{g_0} a_r(P) = a_r(g_0^{-1} P).$$

Hence the functions $a_r(P)$ are transformed under rotations like scalar functions (see §3). The problem of decomposing this representation into irreducible parts is solved by expanding the functions $a_r(\vartheta, \phi)$ in series of spherical functions. It is therefore reasonable to take $a_r(P)$ as one of the components of the vector fields, and we shall do this in the sequel.

We now present a method of describing a vector field with the aid of three functions, *each* of which is transformed under rotations independently of the others.

A function on the sphere is a function of the two variables ϑ and ϕ. The basic idea, which leads quickly to a solution of the problem, is to turn instead to functions of the three variables ϕ_1, θ, ϕ_2 (functions of the rotation g). This makes it possible for us to apply the results of the preceding section.

With this in mind, we consider at the point P on the sphere an arbitrary orthonormal triple of vectors (repère) e_1, e_2, e_3, such that the vector e_3 is directed along the normal to the spherical surface. Every such repère can be defined by a rotation g, which carries the repère located at the north pole with vectors directed along the co-ordinate axes into the given repère. (We shall call this particular repère the normal repère.)

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* This representation is unitary under the following scalar product for two vector functions $a(P)$ and $b(P)$:

$$(a(P), b(P)) = \int_0^{2\pi}\int_0^{\pi} \{a_1(\vartheta, \varphi)\overline{b_1(\vartheta, \varphi)} + a_2(\vartheta, \varphi)\overline{b_2(\vartheta, \varphi)} + a_3(\vartheta, \varphi)\overline{b_3(\vartheta, \varphi)}\} \sin\vartheta \, d\vartheta \, d\varphi.$$

Here ϑ and ϕ are spherical co-ordinates for the point P of the sphere and a_k and b_k are components of a and b in an arbitrary co-ordinate system.

Consider the repère corresponding to the rotation g. Let us subject the vectors of this repère to a certain rotation g_0. Under this, the given repère goes into a new repère. Since the first repère is obtained from the normal repère by the rotation g, and the second from the first by the rotation g_0, it follows that the second is obtained from the normal repère by the rotation g_0g. That is, it is defined by the rotation g_0g.

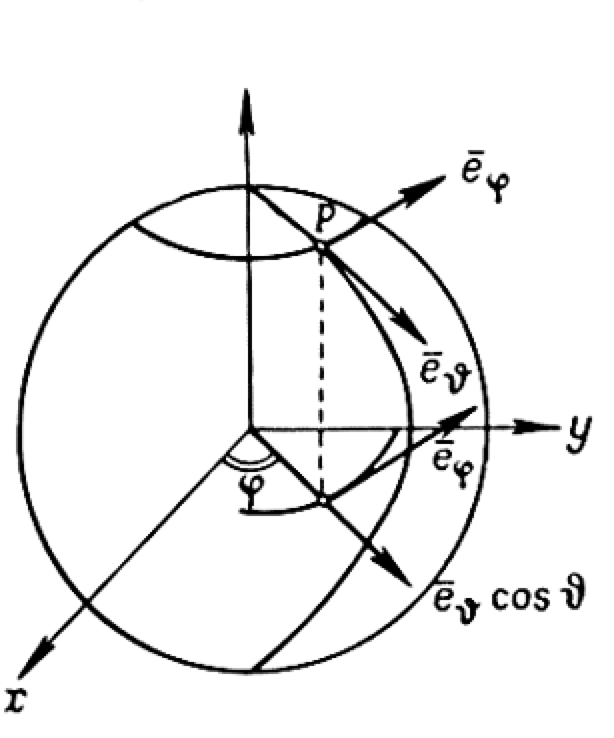

Fig. 5.

As noted above, every repère is defined by a certain rotation g, that is, by the Euler angles ϕ_1, θ, ϕ_2. We shall explain how to describe the repère with the aid of the angles ϕ_1, θ, ϕ_2.

We emphasize that the element g of the rotation group defines not only the position of the repère but also the point P of the sphere which is its origin. The spherical co-ordinates ϑ and ϕ of the point P into which the north pole of the sphere goes under the rotation g are connected with the Euler angles of the rotation g by the relations

$$\vartheta = \theta, \qquad \varphi = \varphi_2 - \frac{\pi}{2}, \tag{3}$$

that is, this point does not depend upon the Euler angle ϕ_1.*

The third vector of the repère, which is normal to the sphere, is completely determined by the position of the origin of the repère and hence does not depend upon the angle ϕ_1.

The vectors e_1 and e_2 lie in the tangent plane to the sphere at the point P and depend upon ϕ_1. In order to determine this dependence, we consider two rotations $g = g(\phi_1, \theta, \phi_2)$ and $g_1 = (\phi_1 + \phi^*, \theta, \phi_2)$ with different values of the first of the Euler angles. It is evident that these rotations carry the normal repère into two repères with origin at one and the same point, and, consequently, with one and the same vector e_3. Also, the second repère (defined by the rotation g_1) is obtained from the first by rotation through the angle ϕ^* in the positive direction around the vector e_3. If e_1 and e_2 are vectors of the first repère and e_1' and e_2' are the corresponding vectors of the second repère, then, according to well-known formulas, we have

$$e_1' = e_1 \cos\varphi^* + e_2 \sin\varphi^*,$$
$$e_2' = -e_1 \sin\varphi^* + e_2 \cos\varphi^*.$$

Let us set $\phi_1 = \pi$, rewrite ϕ^* as ϕ_1, and write $e_k(\pi, \theta, \phi_2)$ as $e_k^0\ (k=1, 2)$. We then obtain

$$\left.\begin{aligned} e_1(\varphi_1, \theta, \varphi_2) &= e_1^0 \cos\varphi_1 + e_2^0 \sin\varphi_1, \\ e_2(\varphi_1, \theta, \varphi_2) &= -e_1^0 \sin\varphi_1 + e_2^0 \cos\varphi_1. \end{aligned}\right\}$$

The vectors $e_1^0 = e_1(\pi, \theta, \phi_2)$ and $e_2^0 = e_2(\pi, \theta, \phi_2)$ have a simple geometric interpretation. They are, in fact, the unit vectors lying along the tangents to the parallel and meridian of the sphere going through the point P. To see this, set $\phi_1 = \pi$ in the matrix $\|g_{ik}\|$. This gives us expressions for the Cartesian co-ordinates of these vectors: $e_1 = (-\cos\phi_2, -\sin\phi_2, 0)$ and $e_2 = (\cos\theta\sin\phi_2, -\cos\theta\cos\phi_2, -\sin\theta)$. On the other hand, it is obvious from Fig, 5 that the unit vector e_ϕ lying along the tangent to the parallel in the direction of increasing ϕ has components $(-\sin\phi, \cos\phi, 0)$. It is also obvious that the unit vector e_ϑ lying along the meridian in the direction of increasing ϑ has components $(\cos\vartheta\cos\phi, \cos\vartheta\sin\phi, -\sin\vartheta)$. Noting that the spherical co-ordinates ϑ and ϕ of the point P which is the origin of the repère are connected with the Euler angles θ and ϕ_2 by the formulas (3), we see that $e_1^0 = -e_\phi$ and $e_2^0 = e_\vartheta$.

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* The Cartesian co-ordinates of the point into which the north pole of the sphere goes under the rotation g are the elements of the third column of the matrix $\|g_{ik}\|$. That is, they are the numbers $\sin\phi_2 \sin\theta$, $-\cos\phi_2 \sin\theta$, $\cos\theta$ (see paragraph 2 of §1). Upon comparing these expressions with expressions for the Cartesian co-ordinates of this point in terms of its spherical co-ordinates, $\cos\phi \sin\vartheta$, $\sin\phi \sin\vartheta$, $\cos\vartheta$, we obtain the formulas (3).

Finally, we obtain

$$\left.\begin{aligned} e_1(\varphi_1\,\theta, \varphi_2) &= -e_\varphi\cos\varphi_1 + e_\vartheta\sin\varphi_1, \\ e_2(\varphi_1\,\theta, \varphi_2) &= e_\varphi\sin\varphi_1 + e_\vartheta\cos\varphi_1, \\ e_3(\varphi_1\,\theta, \varphi_2) &= e_r. \end{aligned}\right\} \tag{4}$$

Here e_r, e_ϕ, and e_ϑ are unit vectors directed, respectively, along the normal to the sphere, and along the tangents to the parallel and meridian of the sphere at the point P whose spherical co-ordinates are $\phi = \phi_2 - \frac{\pi}{2}$ and $\vartheta = \theta$.

We now consider an arbitrary vector function $a(P)$ (that is, a vector field on the sphere). Given a certain repère with origin at the point P, we can write $a(P)$ as a linear combination of the vectors of this repère. This gives us three numbers a_1, a_2, a_3; these are the components of a in terms of the vectors of the repère. These numbers, like the vectors of the repère themselves, are functions of the rotation g, that is, of the Euler angles ϕ_1, θ, ϕ_2. Since e_3 does not depend upon ϕ_1, the component a_3 does not depend upon ϕ_1 either. It coincides with the function $a_r(P)$ defined earlier, computed at the point P with spherical co-ordinates $\phi_2 - \frac{\pi}{2}$ and θ.

The formulas (4) make it clear how the components a_1 and a_2 depend upon ϕ_1. In fact, taking the scalar product of both sides of (4) with $a(P)$, we obtain

$$\left.\begin{aligned} a_1(\varphi_1, \theta, \varphi_2) &= -a_\varphi\cos\varphi_1 + a_\vartheta\sin\varphi_1, \\ a_2(\varphi_1, \theta, \varphi_2) &= a_\varphi\sin\varphi_1 + a_\vartheta\cos\varphi_1, \\ a_3(\varphi_1, \theta, \varphi_2) &= a_r. \end{aligned}\right\} \tag{5}$$

We recall how to express the components a_ϕ and a_ϑ, as well as a_r, in terms of the ordinary components of a, that is, in terms of a_x, a_y, and a_z. For this purpose, we must form the scalar product of the vector $a = (a_x, a_y, a_z)$ with each of the vectors $e_\phi = (-\sin\phi, \cos\phi, 0)$, $e_\vartheta = (\cos\vartheta\cos\phi, \cos\vartheta\sin\phi, -\sin\vartheta)$, and $e_r = (\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta)$. As a result we obtain

$$\left.\begin{aligned} a_\varphi(\varphi, \vartheta) &= -a_x\sin\varphi + a_y\cos\varphi, \\ a_\vartheta(\varphi, \vartheta) &= a_x\cos\vartheta\cos\varphi + a_y\cos\vartheta\sin\varphi - a_z\sin\vartheta, \\ a_r(\varphi, \vartheta) &= a_x\sin\vartheta\cos\varphi + a_y\sin\vartheta\sin\varphi + a_z\cos\vartheta. \end{aligned}\right\} \tag{6}$$

For further calculations, it is convenient to introduce the complex components of the vector a:

$$a_+ = a_1 - ia_2,$$
$$a_- = a_1 + ia_2.$$

From (5), we obtain the following expressions for a_+ and a_- :

$$\left.\begin{aligned} a_+(\varphi_1, \theta, \varphi_2) &= a_+(\pi, \theta, \varphi_2)\, e^{i\varphi_1} = (-a_\varphi - ia_\theta)\, e^{i\varphi_1}, \\ a_-(\varphi_1, \theta, \varphi_2) &= a_-(\pi, \theta, \varphi_2)\, e^{-i\varphi_1} = (-a_\varphi + ia_\theta)\, e^{-i\varphi_1}. \end{aligned}\right\} \tag{7}$$

The components a_ϕ and a are taken at the point with co-ordinates $\phi = \phi_2 - \frac{\pi}{2}$, $\vartheta = \theta'$.

We now examine what happens to the functions a_1, a_2, and a_3 (or, equivalently, to the functions a_+, a_-, and a_3) under rotations. We subject the vector field and the repères under study to the rotation g_0. Since both the vectors of the field $a(P)$ and the vectors of the repères undergo one and the same rotation, it is clear that the components of the rotated vector a' in the new repère are the same as the components of the original vector a in the original repère. The original repère is described with the aid of the rotation g, and the new repère, as we have already seen, with the aid of the rotation $g_0 g$. In this fashion, we see that

$$a'_k(g_0 g) = a_k(g),$$

or, if we write $g_0 g$ as g,

$$a'_k(g) = a_k(g_0^{-1} g). \tag{8}$$

In this fashion, we see that the functions $a_1(g)$, $a_2(g)$, and $a_3(g)$, and consequently $a_+(g)$, $a_-(g)$, and $a_3(g)$, are transformed under rotations independently of each other. That is, the transformations of each of these components generates a certain representation of the rotation group.

We can decompose each of these representations into irreducible parts. This produces expansions for each of the components a_k of the vector $a(P)$ which are invariant under rotations and are independent of each other. This means that they do not get mixed up with each other under rotations.

In §7, we solved the problem of decomposing the regular representation of the rotation group into its irreducible parts. There, however, from considerations of convenience, we used right multiplication of g by g_0 as the transformation corresponding to the rotation g_0, and not left multiplication of g by g_0^{-1}. One can go from one of these transformations to the other without difficulty. For this, it is sufficient to consider in place of the functions $a_k(g)$ the functions $\tilde{a}_k(g) = a_k(g^{-1})$. This leads to replacement of the arguments ϕ_1, θ, ϕ_2 in these functions by $\pi - \phi_2, \theta, \pi - \phi_1$. After this change, the transformation (8) is written in the form

$$\tilde{a}'_k(g) = a'_k(g^{-1}) = a_k(g_0^{-1} g^{-1}) = a_k((g g_0)^{-1}) = \tilde{a}_k(g g_0),$$

that is

$$\tilde{a}'_k(g) = \tilde{a}_k(g g_0). \tag{8'}$$

From this point on, we shall study the functions $\tilde{a}_1$, $\tilde{a}_2$, and $\tilde{a}_3$ of g instead of the functions a_1, a_2, and a_3. Recalling what we know regarding the dependence of the original functions a_1, a_2, and a_3 on the arguments ϕ_1, θ, ϕ_2, and replacing these arguments by $\pi-\phi_2$, θ, $\pi-\phi_1$, respectively, we can assert that the new function $\tilde{a}_3$ does not depend upon ϕ_2 and that

$$\tilde{a}_+(\varphi_1, \theta, \varphi_2) = -e^{-i\varphi_2}\tilde{a}_+(\varphi_1, \theta, 0) = e^{-i\varphi_2}(a_\varphi + i a_\vartheta),$$

$$\tilde{a}_-(\varphi_1, \theta, \varphi_2) = -e^{i\varphi_2}\tilde{a}_-(\varphi_1, \theta, 0) = e^{i\varphi_2}(a_\varphi - i a_\theta).$$

The point P at which we consider the vector a with this components now has spherical co-ordinates $\frac{\pi}{2}-\phi_1$ and θ. It is evident from this what the irreducible representations must be into which the representations for $\tilde{a}_+$, $\tilde{a}_-$, and $\tilde{a}_r$ are decomposed. In fact, the functions $\tilde{a}_r$, since they do not depend upon ϕ_2, can be expanded in series of functions appearing in the zero-th (center) rows of all of the matrices $T_g^l\,(l = 0, 1, 2, \cdots)$; that is, these functions can be expanded in series of spherical functions of the arguments $\frac{\pi}{2}-\phi_1$ and θ. The functions $\tilde{a}_+(\phi_1, \theta, \phi_2)$ contain the factor $e^{-i\phi 2}$ in just the same way as the elements of the first row of the matrices T_g^l (which are the generalized spherical functions $T^l_{1n}(\phi_1, \theta, \phi_2)$), and hence the functions $\tilde{a}_+$ can be expanded in series of just these functions.

In like manner, one sees that the functions $\tilde{a}_-(\phi_1)\,\theta, \phi_2)$ can be expanded in series of spherical functions $T^l_{-1,n}(\phi_1, \theta, \phi_2)$.

We have therefore proved the following assertion. *Let $a(P)$ be a vector function defined on the surface of the unit sphere. An expansion of this function in a series which is invariant under rotations is obtained in the following way. At the point P with spherical co-ordinates ϕ and ϑ, one takes the components of the vector along the parallel, the meridian, and the radius, a_ϕ, a_ϑ, and a_r. From these functions, one defines functions $\tilde{a}_+$, $\tilde{a}_-$ and $\tilde{a}_r$ of three variables by the following formulas:*

$$\left.\begin{aligned}
\tilde{a}_+(\varphi_1, \theta, \varphi_2) &= e^{-i\varphi_2}\Big[\, a_\varphi(\varphi, \theta) + i a_\vartheta(\varphi, \theta)\Big],\\
\tilde{a}_-(\varphi_1, \theta, \varphi_2) &= e^{i\varphi_2}\Big[\, a_\varphi(\varphi, \theta) - i a_\vartheta(\varphi, \theta)\Big],\\
\tilde{a}_r(\varphi_1, \theta, \varphi_2) &= a_r(\varphi, \theta), \text{ where } \varphi = \frac{\pi}{2} - \varphi_1.
\end{aligned}\right\} \tag{9}$$

Each of these functions is expanded in a series of its collection of generalized spherical functions, as follows. The functions $\tilde{a}_+(\phi_1, \theta, \phi_2)$ are expanded in series of the functions $T^l_{1n}(\phi_1, \theta, \phi_2)$ $(l=0, 1, 2, \cdots; -l \le n \le l)$. The functions $\tilde{a}_-(\phi_1, \theta, \phi_2)$ are expanded in series of the functions $T^l_{-1,n}(\phi_1, \theta, \phi_2)$. The functions $a_r(\phi_1, \theta, \phi_2)$ are expanded in series of the functions

$$T^l_{0n}(\varphi_1, \theta, \varphi_2) = \sqrt{\frac{2}{2l+1}}\, Y^l_n\left(\frac{\pi}{2} - \varphi_1, \theta\right).$$

Dropping the common factor $e^{\pm i\phi 2}$ in the first two series and replacing $\frac{\pi}{2}-\phi_1$ everywhere by ϕ, we obtain expansions of the components of the vector $a_\phi \pm i a_\vartheta$ and a_r. These expansions generalize the corresponding expansions of scalar functions given in §3.

2. Expansion of arbitrary quantities. In paragraph 1, we studied the problem of expanding functions whose values at every point are vectors of 3-dimensional space. We now turn to the case in which we have a function $f(x)$ the value of which at every point is a quantity which transforms in conformity with an irreducible representation of weight l_0; that is, we have given a field of the quantity f. This means that at every point x of the space in which the function is defined, we are given $2l_0+1$ numbers c_m $(-l_0 \le m \le l)$, which are the components of the quantity f. Under a rotation g_0, these components undergo the linear transformation

$$c'_m = \sum_{n=-l_0}^{l_0} a_{mn}(g_0)\, c_n. \tag{10}$$

As usual, we shall write the transformation (10) in the form

$$f' = U_{g_0} f. \tag{10'}$$

A *field of quantities* is transformed in the following way under a rotation g_0. First, as the result of the rotation g_0, we take at the point x the value of the quantity f which previously was assumed at the point $g_0^{-1}x$. Second, the quantity undergoes the transformation U_{g_0}. Thus, *under the rotation* g_0, the quantity $f(x)$ is changed into $f'(x) = U_{g_0} f(g_0^{-1}x)$. That is, *we have a transformation* T_{g_0} *of the field of quantities defined by the formula*

$$T_{g_0} f(x) = U_{g_0} f(g_0^{-1}x). \tag{11}$$

(In the case of a vector field, the vectors undergo the same rotation g_0 as the entire space, and hence we found the formula (1) $T_{g_0} a(x) = g_0 a(g_0^{-1}x)$. This is a special case of (11).) As in the case of a vector field, it is evident that these transformations form a representation of the rotation group.

As in paragraph 1, we limit ourselves to fields of quantities defined on the surface of the unit sphere. Thus we write $f(P) = f(\vartheta, \phi)$. We consider the problem of finding an expansion for this field which will be invariant under rotations. This problem is of course analogous to the problem for vector fields solved in the preceding paragraph. As in the case of a vector field, we replace the ordinary components of the quantity by $2l_0+1$ functions of the three variables ϕ_1, θ, ϕ_2 (that is, functions of the rotation g) which define the quantity f and which are transformed independently of each other by rotations. We thus obtain $2l_0+1$ distinct representations of the rotation group, each of which can then easily be decomposed into irreducible parts.

In order to define these new components of the quantity, we make use of the correspondence established in paragraph 1 between rotations g and repères with origin at an arbitrary point of the sphere, the third vector of which lies along the normal to the spherical surface. We can specify the value of the quantity at a given point of the sphere by means of components which refer to different systems of co-ordinates (different repères) in space. Let c_m be the given components of the quantity and let g be a rotation carrying the triple of co-ordinate vectors into the vectors e_1, e_2, e_3. Then the numbers

$$c'_m = \sum_{n=-l_0}^{l_0} a_{mn}(g)\, c_n$$

will be the components referred to the repère e_1, e_2, e_3. (See formula (10).) We shall define the value of the quantity at the point P by its components relative to the repère with origin at the point P and depending upon an arbitrary rotation g carrying the north pole into P. The components of the quantity f at the point P

relative to such a repère will then be certain specified functions of the rotation g, $c_m(g)$ $(-l_0 \leq m \leq l_0)$. These functions are obtained from the original components of the quantity $f(P)$ by the transformation U_g.

We shall now explain how the functions $c_m(g)$ are transformed by an arbitrary rotation g_0. The repère depending upon g is transformed by the rotation g_0 into the repère depending upon $g_0 g$, as we saw in paragraph 1. On the other hand, quantities f and repères are carried from point to point and transformed (rotated) by the rotation g_0 in exactly the same way. Therefore the components of the quantity f in the corresponding basis are not altered by a rotation. We see, in consequence, that

$$c'_m(g_0 g) = c_m(g)$$

or, replacing $g_0 g$ by g,

$$c'_m(g) = c_m(g_0^{-1} g). \tag{12}$$

We see that the functions $c_m(g)$, which serve to define the quantity, are transformed independently of each other by the rotation g_0.

As in paragraph 1, we introduce the functions $\tilde{c}_m(g) = c_m(g^{-1})$ instead of the functions $c_m(g)$. The transformation T_{g_0} for $c_m(g)$ which corresponds to the rotation g_0 is then given, according to (12), by the formula

$$T_{g_0}\tilde{c}_m(g) = \tilde{c}_m(g g_0). \tag{13}$$

This shows that the transformation for the function $\tilde{c}_m(g)$ coincides, for all m, with the transformation studied in §7.

We must decompose each of these representations into irreducible parts. This is done in the simplest possible manner if the functions $\tilde{c}_m(g)$ depend in a special way on the third Euler angle ϕ_2. Namely, we require that

$$\tilde{c}_m(g) = \tilde{c}_m(\varphi_1, \theta, \varphi_2) = e^{-im\varphi_2}\,\tilde{c}_m(\varphi_1, \theta, 0). \tag{14}$$

Functions of this type are expanded in series of elements of the m-th rows of all of the matrices T_g^l, that is, in series of the functions $T_{mn}^l(\phi_1, \theta, \phi_2)$ with a fixed value of m.

Now, if the components of the quantity f have the form (14), we rotate space through the angle ϕ^* around the axis e_3, that is, we add ϕ^* to the angle ϕ_2, and in doing this, we multiply each function $\tilde{c}_m(\phi_1, \theta, \phi_2)$ by $e^{-im\phi^*}$. This implies that the matrix corresponding to a rotation around the axis oz is a diagonal matrix That is, the basis consists of eigenvectors of this matrix.*

We thus obtain the following final result. *To obtain the correct expansion of the field of quantities $f(P)$ defined on the surface of the unit sphere, and transforming under a rotation g_0 according to the formula*

$$T_{g_0} f(P) = U_{g_0} f(g_0^{-1} P), \tag{15}$$

we must proceed in the following way. We first write the quantity $f(P)$ at the point P with spherical co-ordinates ϕ and ϑ in terms of its components $c_m^0(\phi, \vartheta)$ in any basis consisting of eigenvectors of the transformation which corresponds to a rotation around the axis oz. We then go over to the components $c_m(g)$ which depend

---

* It follows from the results of §2 that this basis, except for normalization, coincides with the canonical basis introduced in §2.

upon the rotation $g = g(\phi_1, \theta, \phi_2)$, *where* $\phi_1 = \frac{\pi}{2} - \phi$ *and* $\theta = \vartheta$, *by subjecting the quantity*

$$\left\{ c^0_m\left(\frac{\pi}{2} - \varphi_1, \theta\right)\right\}$$

to the transformation U_g *which corresponds to this rotation. After this, we expand every function* $\tilde{c}_m(g) = \tilde{c}_m(\phi_1, \theta, \phi_2)$ *in a series of the elements of the m-th rows of all the matrices* T^l_g *for all* $l \geq m$. *That is, we set*

$$\tilde{c}_m(\varphi_1, \theta, \varphi_2) = \sum_{l=m}^{\infty} \sum_{n=-l}^{l} a^l_{mn} T^l_{mn}(\varphi_1, \theta, \varphi_2). \tag{16}$$

To conclude this paragraph, we consider the case when the quantity to be expanded is transformed in conformity with an arbitrary representation of the rotation group (not necessarily irreducible, as was assumed above). In this case, the quantity can naturally be decomposed into summands each of which is transformed in conformity with an irreducible representation. All that is necessary is to decompose the representation into irreducible parts. Then one deals with each summand in the manner described above.

However, an even simpler method is at hand. It suffices to describe the quantity by components which are multiplied by $e^{-im\phi}$ under the rotation through the angle ϕ around the axis oz, where m is an integer or a half-integer. One then writes the quantity in spherical co-ordinates (the vectors e_ϕ, e_ϑ, and e_r form a convenient basis). Then one expands the m-th component in a series of functions $T^l_{mn}(\phi_1, \theta, \phi_2)$ ($l \geq m$, $-l \leq n \leq l$). Here if the representation is reducible, a given value of m may be encountered several times (note that this cannot occur with an irreducible representation). This means that different components are expanded in series of the same functions.

3. **Example. A field of tensors of rank 2.** We illustrate the discussion of paragraph 2 by the simple example of a tensor field of rank 2, $a = a_{ij}$. Such a tensor is a quantity which is transformed in conformity with a reducible representation of dimension 9. We have obtained the decomposition of this representation into irreducible parts in §5.

We shall find components of the tensor a_{ij} which are transformed in the necessary way under rotations around the axis oz. For this purpose, we make use of the fact that we know components of a vector a_i which behave in the required way under rotations around oz (see paragraph 1 of the present section). These components are: $a_x - ia_y = a_1 - ia_2$ (this component is multiplied by $e^{-i\phi}$); $a_z = a_3$ (this component is multiplied by 1); $a_x + ia_y = a_1 + ia_2$ (this component is multiplied by $e^{i\phi}$). The components of a tensor of rank 2 are transformed under rotations like products of the components of two vectors.

Therefore we first divide the components of such a tensor into three groups, differing by the effect of a rotation around oz on the first of their indices. These groups are $a_{1j} - ia_{2j}$, a_{3j}, $a_{1j} + ia_{2j}$, ($j = 1, 2, 3$). We then perform the same operation within each group on the second index. We then have 9 complex components for a tensor of rank 2, as follows:

$$
\begin{aligned}
&a_{11}-ia_{21}-i\,(a_{12}-ia_{22})=a_{11}-a_{22}-i\,(a_{21}+a_{12}),\\
&\quad a_{13}-ia_{23},\\
&a_{11}-ia_{21}+i\,(a_{12}-ia_{22})=a_{11}+a_{22}-i\,(a_{21}-a_{12}),\\
&\quad a_{31}-ia_{32},\\
&\qquad a_{33},\\
&\quad a_{31}+ia_{32},\\
&a_{11}+ia_{21}-i\,(a_{21}+ia_{22})=a_{11}+a_{22}+i\,(a_{21}-a_{12}),\\
&\quad a_{13}+ia_{23},\\
&a_{11}+ia_{21}+i\,(a_{12}+ia_{22})=a_{11}-a_{22}+i\,(a_{21}+a_{12}).
\end{aligned}
$$

In order to expand in series of generalized spherical functions, we must subject the tensor at every point P with spherical co-ordinates ϕ, ϑ to the rotation $g = g(\frac{\pi}{2}-\phi_1, \vartheta, 0)$ which carries the point P into the north pole of the sphere.* As a result of this rotation, the basis vectors e_x, e_y, and e_z are carried into $-e_\phi$, e_ϑ, and e_r. We obtain the following 9 components of the tensor, depending on ϕ and ϑ:

a_{rr}, $a_{\phi\phi}+a_{\vartheta\vartheta} \pm i(a_{\phi\vartheta}-a_{\vartheta\phi})$; these components are expanded in series of the 0-th row of the matrices T_g^l (i.e., ordinary spherical functions $Y_l^n(\frac{\pi}{2}-\phi, \vartheta)$);

$-a_{\phi r}-ia_{\vartheta r}$, $-a_{r\phi}-ia_{r\vartheta}$; these components are expanded in series of the functions $T_{1n}^l\left(\frac{\pi}{2}-\varphi, \vartheta, 0\right)$;

$-a_{\phi r}+ia_{\vartheta r}$ and $-a_{r\phi}+ia_{r\vartheta}$; these components are expanded in series of the functions $T_{-1n}^l\left(\frac{\pi}{2}-\varphi, \vartheta, 0\right)$;

$a_{\phi\phi}-a_{\vartheta\vartheta}+i(a_{\vartheta\phi}+a_{\phi\vartheta})$; these components are expanded in series of the functions $T_{2n}^l\left(\frac{\pi}{2}-\varphi, \vartheta, 0\right)$;

$a_{\phi\phi}-a_{\vartheta\vartheta}-i(a_{\vartheta\phi}+a_{\phi\vartheta})$; these components are expanded in series of the functions $T_{-2,n}^l\left(\frac{\pi}{2}-\varphi, \vartheta, 0\right)$.

4. Solution of Maxwell's equations. As an application of the expansions obtained above, we consider the solution of Maxwell's equations.

We shall consider the case in which these equations are written in the form

$$
\ddot{\tilde{A}}-\Delta\tilde{A}=0,
$$
$$
\operatorname{div}\tilde{A}=0,
$$

where $\tilde{A}$ is a vector depending upon x, y, z, t (A is thus the vector potential), and $\Delta\tilde{A}$ is the vector with components $\Delta\tilde{A}_x$, $\Delta\tilde{A}_y$, $\Delta\tilde{A}_z$ and

$$
\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.
$$

Setting $\tilde{A}=A(x, y, z)e^{ikt}$, we obtain the system of equations

$$
\Delta A+k^2A=0,
$$
$$
\operatorname{div}A=0.
$$

Since these equations are invariant with respect to rotations, it is natural to look for a solution in a series of functions which are also invariant under rotations, that is, a series of generalized spherical functions. With this end in view, we go over to spherical co-ordinates and instead of the components A_x, A_y, A_z, we shall define the vector A at every point by the components A_r, A_ϕ, A_ϑ, defined by formulas (6) of the present section.* Carrying out the corresponding transformation of the independent variables and the functions which are being studied, we obtain the following 4 equations:

---

* The inverse rotation g^{-1} must carry the north pole into P (see paragraph 1 of the present section).

$$\frac{\partial^2 A_r}{\partial r^2}+\frac{2}{r}\frac{\partial A_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 A_r}{\partial\vartheta^2}+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 A_r}{\partial\varphi^2}+\frac{\operatorname{ctg}\vartheta}{r^2}\frac{\partial A_r}{\partial\vartheta}-$$
$$-\frac{2}{r^2}\frac{\partial A_\vartheta}{\partial\vartheta}-\frac{2}{r^2\sin\vartheta}\frac{\partial A_\varphi}{\partial\varphi}-\frac{2A_r}{r^2}-\frac{2\operatorname{ctg}\vartheta}{r^2}A_\vartheta+k^2A_r=0, \qquad (17)$$

$$\frac{\partial^2 A_\varphi}{\partial r^2}+\frac{2}{r}\frac{\partial A_\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 A_\varphi}{\partial\vartheta^2}+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 A_\varphi}{\partial\varphi^2}+\frac{\operatorname{ctg}\vartheta}{r^2}\frac{\partial A_\varphi}{\partial\vartheta}+$$
$$+\frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial A_\vartheta}{\partial\varphi}+\frac{2}{r^2\sin\vartheta}\frac{\partial A_r}{\partial\varphi}-\frac{A_\varphi}{r^2\sin^2\vartheta}+k^2A_\varphi=0,$$

$$\frac{\partial^2 A_\vartheta}{\partial r^2}+\frac{2}{r}\frac{\partial A_\vartheta}{\partial r}+\frac{1}{r^2}\frac{\partial A_\vartheta}{\partial\vartheta^2}+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 A_\vartheta}{\partial\varphi^2}+\frac{\operatorname{ctg}\vartheta}{r^2}\frac{\partial A_\vartheta}{\partial\vartheta}-$$
$$-\frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial A_\varphi}{\partial\varphi}+\frac{2}{r^2}\frac{\partial A_r}{\partial\vartheta}-\frac{A_\vartheta}{r^2\sin^2\vartheta}+k^2A_\vartheta=0,$$

$$\frac{\partial A_r}{\partial r}+\frac{1}{r}\frac{\partial A_\vartheta}{\partial\vartheta}+\frac{1}{r\sin\vartheta}\frac{\partial A_\varphi}{\partial\varphi}+\frac{2}{r}A_r+\frac{\operatorname{ctg}\vartheta}{r}A_\vartheta=0.$$

We now set down the combinations of the components which are expanded in generalized spherical functions, as follows:

$$A_+=-\frac{1}{\sqrt{2}}(A_\varphi+iA_\vartheta),$$
$$A_0=A_r,$$
$$A_-=\frac{1}{\sqrt{2}}(A_\varphi-iA_\vartheta).$$

In these components, equations (17) assume the form

$$\frac{\partial^2 A_0}{\partial r^2}+\frac{2}{r}\frac{\partial A_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 A_0}{\partial\vartheta^2}+\frac{1}{r^2}\operatorname{ctg}\vartheta\frac{\partial A_0}{\partial\vartheta}+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 A_0}{\partial\varphi^2}-\frac{2}{r^2}A_0+k^2A_0-$$
$$-\frac{i\sqrt{2}}{r^2}\left[\frac{\partial A_+}{\partial\vartheta}+\frac{i}{\sin\vartheta}\frac{\partial A_+}{\partial\varphi}+\operatorname{ctg}A_+\right]-\frac{i\sqrt{2}}{r^2}\left[\frac{\partial A_-}{\partial\vartheta}-\frac{i}{\sin\vartheta}\frac{\partial A_-}{\partial\varphi}+\operatorname{ctg}\vartheta A_-\right]=0,$$

$$\frac{\partial^2 A_-}{\partial r^2}+\frac{2}{r}\frac{\partial A_-}{\partial r}+\frac{1}{r^2}\frac{\partial^2 A_-}{\partial\vartheta^2}+\frac{1}{r^2}\operatorname{ctg}\vartheta\frac{\partial A_-}{\partial\vartheta}+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 A_-}{\partial\varphi^2}+\frac{2i\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial A_-}{\partial\varphi}-$$
$$-\frac{1}{r^2\sin^2\vartheta}A_-+k^2A_--\frac{i\sqrt{2}}{r^2}\left(\frac{\partial A_0}{\partial\vartheta}+i\frac{\partial A_0}{\partial\varphi}\right)=0, \qquad (17')$$

$$\frac{\partial^2 A_+}{\partial r^2}+\frac{2}{r}\frac{\partial A_+}{\partial r}+\frac{1}{r^2}\frac{\partial^2 A_+}{\partial\vartheta^2}+\frac{1}{r^2}\operatorname{ctg}\vartheta\frac{\partial A_+}{\partial\vartheta}+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 A_+}{\partial\varphi^2}-\frac{2i\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial A_+}{\partial\varphi}-$$
$$-\frac{1}{r^2\sin^2\vartheta}A_++k^2A_+-\frac{i\sqrt{2}}{r^2}\left(\frac{\partial A_0}{\partial\vartheta}-i\frac{\partial A_0}{\partial\varphi}\right)=0,$$

$$\frac{\partial A_0}{\partial r}+\frac{2}{r}A_0+\frac{i}{r\sqrt{2}}\left(\frac{\partial A_+}{\partial\vartheta}+\frac{i}{\sin\vartheta}\frac{\partial A_+}{\partial\varphi}+\operatorname{ctg}\vartheta A_+\right)+$$
$$+\frac{i}{r\sqrt{2}}\left(\frac{\partial A_-}{\partial\vartheta}-\frac{i}{\sin\vartheta}\frac{\partial A_-}{\partial\varphi}+\operatorname{ctg}\vartheta A_-\right)=0.$$

---

* It is evident that the letter "*a*" is to be replaced by "*A*" throughout the formulas (6).

We shall look for a solution of the system (17′) in the form of series of generalized spherical functions

$$A_0(r, \varphi, \vartheta) = \sum_{l=0}^{\infty} f_l^0(r) \sum_{n=-l}^{l} \alpha_{l,n} T_{0n}^l\left(\frac{\pi}{2} - \varphi, \vartheta, 0\right),$$

$$A_+(r, \varphi, \vartheta) = \sum_{l=0}^{\infty} f_l^+(r) \sum_{n=-l}^{l} \beta_{l,n} T_{1,n}^l\left(\frac{\pi}{2} - \varphi, \vartheta, 0\right),$$

$$A_-(r, \varphi, \vartheta) = \sum_{l=0}^{\infty} f_l^-(r) \sum_{n=-l}^{l} \gamma_{l,n} T_{-1,n}^l\left(\frac{\pi}{2} - \varphi, \vartheta, 0\right).$$

We see here the advantage of the invariant method of solving the problem. This advantage lies in the fact that the variables are now separated, and we need only solve a system of ordinary differential equations in order to determine the functions of r.

In view of the linearity of the equations, it is possible to substitute in the system each individual summand of the series. That is, we may substitute the functions

$$\left.\begin{aligned} A_0(r, \varphi, \vartheta) &= f_l^0(r) T_{0n}^l\left(\frac{\pi}{2} - \varphi, \vartheta, 0\right), \\ A_+(r, \varphi, \vartheta) &= f_l^+(r) T_{1n}^l\left(\frac{\pi}{2} - \varphi, \vartheta, 0\right), \\ A_-(r, \varphi, \vartheta) &= f_l^-(r) T_{-1,n}^l\left(\frac{\pi}{2} - \varphi, \vartheta, 0\right). \end{aligned}\right\} \tag{18}$$

After this substitution, the first of the equations (17′) assumes the form

$$\begin{aligned} &\left[\frac{d^2 f_l^0}{dr^2} + \frac{2}{r}\frac{df_l^0}{dr} + k^2 f_l^0(r)\right] T_{0n}^l\left(\frac{\pi}{2} - \varphi, \vartheta, 0\right) + \\ &\quad + \frac{1}{r^2} f_l^0(r)\left[\frac{\partial^2 T_{0n}^l}{\partial\vartheta^2} + \operatorname{ctg}\vartheta\frac{\partial T_{0n}}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2 T_{0n}}{\partial\varphi^2} - \frac{2}{r^2} T_{0n}\right] \\ &\quad - \frac{i\sqrt{2}}{r^2} f_l^+(r)\left[\frac{\partial T_{1,n}}{\partial\vartheta} + \frac{i}{\sin\vartheta}\frac{\partial T_{1,n}}{\partial\varphi} + \operatorname{ctg}\vartheta T_{1,n}\right] - \\ &\quad - \frac{i\sqrt{2}}{r^2} f_l^-(r)\left[\frac{\partial T_{-1,n}}{\partial\vartheta} - \frac{i}{\sin\vartheta}\frac{\partial T_{-1,n}}{\partial\varphi} + \operatorname{ctg}\vartheta T_{-1,n}\right] = 0. \end{aligned} \tag{19}$$

We now recall that $T_{mn}\left(\frac{\pi}{2} - \varphi, \vartheta, 0\right) = e^{-in\left(\frac{\pi}{2} - \varphi\right)} u_{mn}(\vartheta)$. We also use the differential equation for $u_{mn}(\vartheta)$ (see §7):

$$\frac{d^2 u_{0n}}{d\vartheta^2} + \operatorname{ctg}\vartheta\frac{du_{0n}}{d\vartheta} + \left[l(l+1) - \frac{n^2}{\sin^2\vartheta}\right] u_{0n}(\vartheta) = 0.$$

We also use the recurrence formulas (see the appendix to §7)

$$\frac{du_{1,n}}{d\vartheta} - \frac{n - \cos\vartheta}{\sin\vartheta} u_{1,n} = -i\sqrt{l(l+1)}\, u_{0n},$$

$$\frac{du_{-1,n}}{d\vartheta} + \frac{n + \cos\vartheta}{\sin\vartheta} u_{-1,n} = -i\sqrt{l(l+1)}\, u_{0n}.$$

With the aid of these relations, we can eliminate

$$\frac{d^2 u_{0n}}{d\vartheta^2}, \quad \frac{du_{0n}}{d\vartheta}, \quad \frac{du_{1n}}{d\vartheta}, \quad u_{1n}, \quad \frac{du_{-1,n}}{d\vartheta}, \quad u_{-1,n}$$

from the equation (19). We then cancel out the factor $e^{-in(\frac{\pi}{2}-\phi)}u_{0n}(\vartheta)$ from this equation. We then obtain an equation containing only functions of r:

$$\frac{d^2f_l^0}{dr^2}+\frac{2}{r}\frac{df_l^0}{dr}+\left[k^2-\frac{2+l(l+1)}{r^2}\right]f_l^0(r)-\frac{\sqrt{2}\,\sqrt{l(l+1)}}{r^2}\left[f_l^+(r)+f_l^-(r)\right]=0. \quad (20)$$

We thus see that actually our substitution has led to a separation of variables in equation (19). The analogous substitution and transformation of the two following equations of the system (18) give the two following ordinary differential equations:

$$\left.\begin{aligned}\frac{d^2f_l^+}{dr^2}+\frac{2}{r}\frac{df_l^+}{dr}+\left[k^2-\frac{l(l+1)}{r^2}\right]f_l^+(r)-\frac{\sqrt{2}\,\sqrt{l(l+1)}}{r^2}f_l^0(r)=0,\\ \frac{d^2f_l^-}{dr^2}+\frac{2}{r}\frac{df_l^-}{dr}+\left[k^2-\frac{l(l+1)}{r^2}\right]f_l^-(r)-\frac{\sqrt{2}\,\sqrt{l(l+1)}}{r^2}f_l^0(r)=0.\end{aligned}\right\} \quad (21)$$

The equation $\operatorname{div} A=0$, after the corresponding transformations, assumes the form

$$\frac{df_l^0}{dr}+\frac{2}{r}f_l^0(r)+\frac{\sqrt{l(l+1)}}{r\sqrt{2}}\left[f_l^-(r)+f_l^+(r)\right]=0. \quad (22)$$

It now remains to solve the system of equations (29), (21), (22). With this end in view, we first eliminate the function $f_l^-(r)+f_l^+(r)$ from equations (21) and (22). We then obtain the following differential equation of the second order for $f_l^0(r)$:

$$\frac{d^2f_l^0}{dr^2}+\frac{4}{r}\frac{df_l^0}{dr}+\left[k^2+\frac{2-l(l+1)}{r^2}\right]f_l^0(r)=0. \quad (23)$$

By a change of the unknown function, this equation is made into Bessel's equation of order $l+\frac{1}{2}$. Therefore the solution of equation (23) which is bounded at zero has the form

$$f_l^0(r)=C_1\frac{J_{l+\frac{1}{2}}(kr)}{r^{\frac{3}{2}}}. \quad (24)$$

We obtain from equation (22)

$$f_l^+(r)+f_l^-(r)=-\frac{C_1\sqrt{2}}{\sqrt{l(l+1)}}\left[\frac{1}{2}\frac{J_{l+\frac{1}{2}}(kr)}{r^{\frac{3}{2}}}+\frac{kJ'_{l+\frac{1}{2}}(kr)}{r^{\frac{1}{2}}}\right]. \quad (25)$$

Subtracting one of the equations (21) from the other and denoting $f_l^+(r)-f_l^-(r)$ by $\phi(r)$, we obtain the following equation for $\phi(r)$:

$$\frac{d^2\varphi}{dr^2}+\frac{2}{r}\frac{d\varphi}{dr}+\left[k^2-\frac{l(l+1)}{r^2}\right]\varphi(r)=0.$$

This gives us

$$\varphi(r)=f_l^+(r)-f_l^-(r)=C_2\frac{J_{l+\frac{1}{2}}(r)}{r^{\frac{1}{2}}}. \quad (26)$$

Solving equations (25) and (26), we find

$$f_l^+(r)=\frac{1}{2}\left\{-\frac{C_1}{\sqrt{2}\sqrt{l(l+1)}}\frac{J_{l+\frac{1}{2}}(kr)}{r^{\frac{3}{2}}}-kC_1\frac{\sqrt{2}}{\sqrt{l(l+1)}}\frac{J'_{l+\frac{1}{2}}(kr)}{r^{\frac{1}{2}}}+C_2\frac{J_{l+\frac{1}{2}}(kr)}{r^{\frac{1}{2}}}\right\},$$

$$f_l^-(r)=\frac{1}{2}\left\{-\frac{C_1}{\sqrt{2}\sqrt{l(l+1)}}\frac{J_{l+\frac{1}{2}}(kr)}{r^{\frac{3}{2}}}-kC_1\frac{\sqrt{2}}{\sqrt{l(l+1)}}\frac{J'_{l+\frac{1}{2}}(kr)}{r^{\frac{1}{2}}}-C_2\frac{J_{l+\frac{1}{2}}(kr)}{r^{\frac{1}{2}}}\right\}.$$

Using the recurrence formulas for Bessel's functions, we can replace $J'_{l+½}(kr)$ by $\frac{l+½}{kr}J_{l+½}(kr)-J_{l+3/2}(kr)$. We can thus write these solutions in the form of linear combinations of Bessel's functions

$$f_l^+(r)=\frac{1}{2}\left\{\frac{kC_1\sqrt{2}}{\sqrt{l(l+1)}}\frac{J_{l+\frac{3}{2}}(kr)}{r^{\frac{1}{2}}}-\frac{C_1\sqrt{2}}{\sqrt{l(l+1)}}\frac{(l+1)}{r^{\frac{3}{2}}}J_{l+\frac{1}{2}}(kr)+C_2\frac{J_{l+\frac{1}{2}}(kr)}{r^{\frac{1}{2}}}\right\},$$

$$f_l^-(r)=\frac{1}{2}\left\{\frac{kC_1\sqrt{2}}{\sqrt{l(l+1)}}\frac{J_{l+\frac{3}{2}}(kr)}{r^{\frac{1}{2}}}-\frac{C_1\sqrt{2}}{\sqrt{l(l+1)}}\frac{(l+1)}{r^{\frac{3}{2}}}J_{l+\frac{1}{2}}(kr)-C_2\frac{J_{l+\frac{1}{2}}(kr)}{r^{\frac{1}{2}}}\right\},$$

$$f_l^0(r)=C_1\frac{J_{l+\frac{1}{2}}(kr)}{r^{\frac{3}{2}}}.$$

Accordingly, for every l and n, we obtain a solution depending upon two arbitrary constants C_1 and C_2.

Substituting these functions in the formulas (18), we obtain particular solutions of Maxwell's equations. An arbitrary solution can be represented as a series of such particular solutions.

First setting $C_1 = 1$, $C_2 = 0$ and then $C_1 = 0$, $C_2 = 1$, we find two solutions, which have different physical significance. They differ in their behavior under reflection in the origin of co-ordinates.* The first of the solutions ($C_2 = 0$) is multiplied by $(-1)^l$ under reflection in the origin and bears the name of vector potential of the *electrical multipole* of order l. The second solution ($C_1 = 0$) is multiplied by $(-1)^{l+1}$ and is called the vector potential of the *magnetic multipole* of order l.

By using the recurrence formulas which express T^l_{1n} and $T^l_{-1,n}$ by means of T^{l-1}_{0n}, T^l_{0n}, and T^{l+1}_{0n} (see formula (8) of the appendix to §7), we can express the solution so obtained in terms of the ordinary spherical functions of order $l-1$, l, and $l+1$.

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* In order to verify what happens to A_+, A_- and A_0 under reflection in the origin, we note that under reflection in the origin, the points with co-ordinates r, ϕ, ϑ goes into the point with co-ordinates r, $\phi+\pi$, $\pi-\vartheta$. The components A_r, A_ϕ, A_ϑ go into A_r, A_ϕ, $-A_\vartheta$. Furthermore,

$$e^{-in\left(\frac{\pi}{2}-\varphi+\pi\right)}=(-1)^n e^{-in\left(\frac{\pi}{2}-\varphi\right)}$$

and $u^l_{mn}(\pi-\vartheta)=(-1)^{m+n+l}u^l_{mn}(\vartheta)$.

§9. Equations which are invariant under rotations

In the present section, we shall study systems of partial differential equations which are invariant under rotations of space. Let $\psi_1(x_1, x_2, x_3)$, $\psi_2(x_1, x_2, x_3), \cdots$, $\psi_N(x_1, x_2, x_3)$ be unknown functions. We shall denote the collection of all of these functions by $\psi(x_1, x_2, x_3)$. We write the system itself in the form

$$L_1 \frac{\partial \psi}{\partial x_1} + L_2 \frac{\partial \psi}{\partial x_2} + L_3 \frac{\partial \psi}{\partial x_3} + \varkappa \psi = 0, \tag{1}$$

where L_1, L_2, L_3 are matrices of the N-th order and $\varkappa$ is a number.*

In order to define what we mean by invariance of the equation (1) under rotations, we must state the law according to which the functions $\psi_1, \psi_2, \cdots, \psi_N$ are transformed under rotations. Since the product of rotations must correspond to the successive carrying out of the corresponding transformations on the functions ψ_1, $\psi_2, \cdots, \psi_N$, it follows that the quantity ψ must be transformed in conformity with some specific representation of the rotation group (in general, this representation is reducible). In this way, the quantity goes into $\psi' = T_g \psi$ under the rotation g.

The system of equations (1) *is said to be invariant under rotations if under the transformation $x' = gx$ of the independent variables (g being an arbitrary rotation) and under the corresponding transformation $\psi' = T_g \psi$ of the unknown functions, the system is not altered.* In this section, we find the general form of equations of the first order which are invariant under rotations. Also, with the aid of expansions in series of generalized spherical functions, we reduce the solution of this system of equations to the solution of a system of ordinary differential equations of the first order, having the same number of equations and unknown functions, and admitting solutions which are linear combinations of cylinder functions.

1. Definition of invariant equations. In order to determine the form of invariant equations, we first set down the condition of invariance of the system (1). We subject space to the rotation $g : x' = gx$, that is, we make a change of the independent variables $x_i' = \Sigma_{k=1}^{3} g_{ik} x_k$. Instead of ψ we must also write $\psi' = T_g \psi$ in the equation. Replacing ψ by $T_g^{-1}\psi'$, and replacing differentiation with respect to x_k by differentiation with respect to x_i' according to the formula

$$\frac{\partial}{\partial x_k} = \sum g_{ik} \frac{\partial}{\partial x_i'} ,$$

we obtain the system of equations

$$\sum_{i=1}^{3} \left[g_{i1} L_1 \frac{\partial (T_g^{-1} \psi')}{\partial x_i'} + g_{i2} L_2 \frac{\partial (T_g^{-1} \psi')}{\partial x_i'} + g_{i3} L_3 \frac{\partial (T_g^{-1} \psi')}{\partial x_i'} \right] + \varkappa T_g^{-1} \psi' = 0.$$

Since T_g is a constant matrix, we can rewrite this as

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* The general form of a system of equations of the first order is

$$A_1 \frac{\partial \psi}{\partial x_1} + A_2 \frac{\partial \psi}{\partial x_2} + A_3 \frac{\partial \psi}{\partial x_3} + B\psi = 0.$$

If the matrix B is not singular, then, applying the matrix B^{-1} to both sides of this equation, we obtain a system of the form (1) (with $\varkappa = 1$). In just the same way, upon applying the matrix CB^{-1} to the system, we can produce a matrix equal to C which is applied to the function ψ. We note that applying a non-singular matrix to the system does not alter the system, since it is equivalent to replacing the given equations by linear combinations of these equations.

$$\sum_i \left[g_{i1}L_1T_g^{-1}\frac{\partial\psi'}{\partial x_i'} + g_{i2}L_2T_g^{-1}\frac{\partial\psi'}{\partial x_i'} + g_{i3}L_3T_g^{-1}\frac{\partial\psi'}{\partial x_i'}\right] + \varkappa T_g^{-1}\psi' = 0.$$

In order for this last system of equations to coincide with the system (1), we first make the coefficient of ψ' equal to $\varkappa$, by applying the transformation T_g to the last system of equations. This gives us the system

$$\sum_k \sum_i g_{ik}T_gL_kT_g^{-1}\frac{\partial\psi'}{\partial x_i'} + \varkappa\psi' = 0. \tag{2}$$

The requirement of invariance implies, consequently, that the relations

$$\sum_k g_{ik}T_gL_kT_g^{-1} = L_i \tag{3}$$

must hold among the matrices L_k for all rotations g. We can write the condition of invariance in a different form, which is sometimes more convenient. For this purpose, we consider the matrix ΣL_ip_i, where p_1, p_2, p_3 are components of a certain vector. Since the matrix L_i is changed into $\Sigma g_{ik}L_k$ under a rotation, it follows that ΣL_ip_i goes into $\Sigma L_kp_k'$, where $p_k' = \Sigma g_{ik}p_i$. That is, the numbers p_i are transformed under rotations just like the partial derivatives $\partial/\partial x_i$.

The equality

$$\sum_{i=1}^{3} L_ip_i = \sum_{k=1}^{3}\sum_{i=1}^{3} g_{ik}p_iT_gL_kT_g^{-1} = \sum_{k=1}^{3} T_gL_kp_k'T_g^{-1} = T_g\left(\sum L_kp_k'\right)T_g^{-1},$$

forms from formula (3). That is,

$$\sum_{i=1}^{3} L_ip_i = T_g\left(\sum_{k=1}^{3} L_kp_k'\right)T_g^{-1}. \tag{4}$$

Equality (4), like (3), is a way of writing the condition of invariance of the system (1) under rotations. With the aid of formula (4), we can immediately find the characteristic polynomial of an arbitrary invariant system. As a matter of fact, we can first form the determinant of both sides of the equality (4). Since $\det T_g = \dfrac{1}{\det T_g^{-1}}$, we see that

$$\det\sum L_ip_i = \det\sum L_kp_k',$$

that is, the characteristic polynomial is not changed by rotations.* Since any two vectors of the same length can be carried one into the other by a certain rotation, it follows that this function is constant on the surface of every sphere with center at the origin and hence depends only on $r = \sqrt{p_1^2 + p_2^2 + p_3^2}$. But $\det\Sigma L_ip_i$ is a homogeneous function of degree N, where N is the number of equations and unknown functions in the system. Consequently, *the characteristic polynomial of the system is equal to*

$$C\,(p_1^2 + p_2^2 + p_3^2)^{N/2}.$$

Since $\det\Sigma L_ip_i$ is clearly a rational function of p_1, p_2, p_3 the number $N/2$ must be an integer, and hence *the number of equations and unknown functions in an invariant system must be even.*

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* By the characteristic polynomial of the system (1), we mean the polynomial in p_i which is equal to $\det\Sigma L_ip_i$.

2. A restatement of the conditions for invariance. The conditions (3) are actually an infinite number of equalities, since the rotation g which enters in (3) is quite arbitrary. We shall now replace these equalities by a finite number of algebraic relations. In order to obtain these relations, we replace the rotation g by infinitesimal rotations around each of the co-ordinate axes.

First, we set the rotation g equal to $e + a_1\xi + \cdots$, where a_1 is an infinitesimal rotation around the axis ox (see §2). The matrix of this rotation g, up to terms of order higher than the first with respect to ξ, is

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & -\xi \\ 0 & \xi & 1 \end{Vmatrix}.$$

The corresponding transformation T_g is $E + \xi A_1 + \cdots$, where A_1 is the transformation corresponding to the infinitesimal rotation a_1. (We defined these transformations in §2.) The inverse transformation, up to terms of order higher than the first, is equal to $E - \xi A_1$.

Substituting $g = e + a_1\xi + \cdots$, $T_g + E + \xi A_1 + \cdots$, $T_g^{-1} = E - \xi A_1 + \cdots$ in the system (3), we obtain, up to terms of order higher than the first, the following three equalities:

$$(E + A_1\xi)\, L_1\, (E - A_1\xi) = L_1,$$
$$(E + A_1\xi)(L_2 - \xi L_3)(E - A_1\xi) = L_2,$$
$$(E + A_1\xi)(\xi L_2 + L_3)(E - A_1\xi) = L_3.$$

Multiplying out, we find, as is to be expected, that the terms not involving ξ all cancel each other. Setting equal to zero the summands which contain the first power of ξ, we obtain the equalities

$$A_1L_1 - L_1A_1 = 0,$$
$$A_1L_2 - L_2A_1 - L_3 = 0,$$
$$A_1L_3 - L_3A_1 + L_2 = 0.$$

These can be written in abbreviated form as

$$[A_1, L_1] = 0,$$
$$[A_1, L_2] = L_3,$$
$$[A_1, L_3] = -L_2.$$

Thus, setting $g = e + a_1\xi + \cdots$, we find the commutators of all three matrices L_1, L_2, L_3 with the known matrix A_1.

Analogously, setting $g = e + a_2\xi + \cdots$ and $g = e + a_3\xi + \cdots$, we obtain the commutators of the matrices L_1, L_2, L_3 with A_2 and A_3. The result can be written in the form of the following table of equations:

$$\begin{array}{lll} [A_1, L_1] = 0, & [A_1, L_2] = L_3, & [A_1, L_3] = -L_2, \\ [A_2, L_1] = -L_3, & [A_2, L_2] = 0, & [A_2, L_3] = L_1, \\ [A_3, L_1] = L_2, & [A_3, L_2] = -L_1, & [A_3, L_3] = 0. \end{array} \qquad (5)$$

From these equations, we determine the possible form of the matrices L_1, L_2, and L_3.

We shall not prove here that equalities (3) and (4) follow from the equalities (5). This proof, which is necessary for the present argument, shows that the validity of certain assertions for infinitesimal rotations implies their validity for all finite rotations. The proof is completely analogous to the proof given in §2 of the fact that the matrices corresponding to infinitesimal rotations complete determine the transformation T_g for all g.

In order to find L_1, L_2, and L_3, we first eliminate the matrices L_1 and L_2 from the system (5). For this purpose, we use the transformations (already introduced in §2)

$$H_+ = H_1 + iH_2 = iA_1 - A_2,$$
$$H_- = H_1 - iH_2 = iA_1 + A_2.$$

We then compute the commutator $[[L_3, H_-], H_+]$. We first find the commutator of L_3 and H_-. Using the equalities (5), we obtain

$$[L_3, H_-] = i[L_3, A_1] + [L_3, A_2] = iL_2 - L_1.$$

We now form the commutator of this operator with H_+. This gives us

$$[iL_2 - L_1, H_+] = [iL_2 - L_1, iA_1 - A_2] = -[L_2, A_1] + [L_1, A_2] = 2L_3.$$

In this way, we obtain two equations:

$$\left.\begin{array}{l}[L_3, H_3] = 0,\\ [[L_3, H_-], H_+] = 2L_3,\end{array}\right\} \qquad (6)$$

The matrix L_3 must satisfy both of these equations.

It can be shown that if we have a matrix L_3 which satisfies the equations (6), and if $L_1 = [A_2, L_3]$ and $L_2 = -[A_1, L_3]$, then the matrices L_1, L_2, L_3 satisfy all the equations of the system (5).

3. **Determination of the matrices L_1, L_2, L_3.** In this paragraph, we shall determine explicitly the matrices L_3 which are solutions of the system (6), and then we shall find L_1 and L_2. The quantity ψ is transformed by some representation, which we shall suppose to be decomposed into irreducible parts. We shall number the components of ψ by (possibly repeated) indices l and m, where l is the weight of the irreducible representation, and m is the number of the component in the representation of weight l. If the irreducible representation with a given weight l is encountered more than once in the decomposition of the representation in ψ into irreducible parts, then we shall add another index τ, which shows the number of the representation of weight l. Thus, when we use these components, the quantity ψ will be written thus:

$$\psi(x_1, x_2, x_3) = \{\psi_{lm}^{\tau}(x_1, x_2, x_3)\}.$$

Let $\xi_{l_0 m_0}^{\tau 0}$ be the quantity for which $\psi_{l_0 m_0}^{\tau 0} = 1$ and all of the other components are zero. We can then write

$$\psi(x_1, x_2, x_3) = \sum_{l, m, \tau} \psi_{lm}^{\tau}(x_1, x_2, x_3)\, \xi_{lm}^{\tau}. \; *$$

Since the quantity ξ_{lm}^{τ} is defined by three indices l, m, τ, a transformation of this quantity, in particular the matrix L_3, is defined by 6 indices. The transformation L_3 of the vectors ξ_{lm}^{τ} therefore has the form

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[*] Thus the quantities ξ_{lm}^{τ} form a basis in the space in which the representation $g \to T_g$ operates. The numbers ψ_{lm}^{τ} are the components of a quantity in this basis.

$$L_3\xi^{\tau}_{lm} = \sum c^{\tau'\tau}_{l'l,\, m'm}\, \xi^{\tau'}_{l'm'}.$$

In order to find the numbers $c^{\tau'\tau}_{l'l,m'm}$, we use the system (6). Recalling from §2 that

$$H_3\xi^{\tau}_{lm} = m\xi^{\tau}_{lm}, \quad H_+\xi^{\tau}_{lm} = \alpha^l_{m+1}\, \xi^{\tau}_{l,\, m+1}, \quad H_-\xi^{\tau}_{lm} = \alpha^l_m\, \xi^{\tau}_{l,m-1},$$

where $(a^l_m)^2 = (l+m)(l-m+1)$, we have

$$L_3H_3\xi^{\tau}_{lm} = mL_3\xi^{\tau}_{lm} = m \sum_{l',\, \tau',\, m'} c^{\tau'\tau}_{l'l,\, m'm}\, \xi^{\tau'}_{l'm'},$$

$$H_3L_3\xi^{\tau}_{lm} = H_3 \sum c^{\tau'\tau}_{l'l,\, m'm}\, \xi^{\tau'}_{l'm'} = \sum_{l'm'\tau'} m' c^{\tau'\tau}_{l'l,\, m'm}\, \xi^{\tau'}_{l'm'}.$$

Since the first of the equations (6) implies that $(L_3H_3 - H_3L_3)\xi^{\tau}_{lm} = 0$, it follows further that

$$\sum_{l',\, m',\, \tau'} (m-m')\, c^{\tau'\tau}_{l'l,\, m'm}\, \xi^{\tau'}_{l'm'} = 0.$$

Setting the coefficients of $\xi^{\tau'}_{l'm'}$ equal to zero, we find that the coefficients $c^{\tau'\tau}_{l'l,m'm}$ can be different from zero only for $m' = m$. For the sake of brevity, we write $c^{\tau'\tau}_{l'l,mm}$ as $c^{\tau'\tau}_{l'l,m}$.

We now make use of the second equation of the system (6). This gives us

$$L_3H_-\xi^{\tau}_{lm} = L_3\alpha^l_m\, \xi^{\tau}_{l,\, m-1} = \alpha^l_m \sum_{l'\tau'} c^{\tau'\tau}_{l'l,\, m-1}\, \xi^{\tau'}_{l',\, m-1},$$

$$H_-L_3\xi^{\tau}_{lm} = H_- \sum_{l'\tau'} c^{\tau'\tau}_{l'l,\, m}\, \xi^{\tau'}_{l',\, m} = \sum_{l'\tau'} \alpha^{l'}_m\, c^{\tau'\tau}_{l'l,\, m}\, \xi^{\tau'}_{l',\, m-1},$$

$$[L_3, H_-]\, \xi^{\tau}_{lm} = \sum_{l'\tau'} [\alpha^l_m\, c^{\tau'\tau}_{l'l,\, m-1} - \alpha^{l'}_m\, c^{\tau'\tau}_{l'l,\, m}]\, \xi^{\tau'}_{l',\, m-1}.$$

Furthermore,

$$[L_3, H_-]\, H_+\xi^{\tau}_{lm} = [L_3, H_-]\, \alpha^l_{m+1}\, \xi^{\tau}_{l,\, m+1} = \alpha^l_{m+1} \sum_{l'\tau'} [\alpha^l_{m+1}\, c^{\tau'\tau}_{l'l,\, m} - \alpha^{l'}_{m+1}\, c^{\tau'\tau}_{l'l,\, m+1}]\, \xi^{\tau'}_{l,m},$$

$$H_+[L_3, H_-]\, \xi^{\tau}_{lm} = H_+ \sum_{l'\tau'} [\alpha^l_m\, c^{\tau'\tau}_{l'l,\, m-1} - \alpha^{l'}_m\, c^{\tau'\tau}_{l'l,\, m}]\, \xi^{\tau'}_{l',\, m-1} =$$

$$= \sum_{l'\tau'} \alpha^{l'}_m [\alpha^l_m\, c^{\tau'\tau}_{l'l,\, m-1} - \alpha^{l'}_m\, c^{\tau'\tau}_{l'l,\, m}]\, \xi^{\tau'}_{l'm},$$

$$[[L_3, H_-], H_+]\, \xi^{\tau}_{lm} =$$

$$= \sum_{l'\tau'} \{[(\alpha^l_{m+1})^2 + (\alpha^{l'}_m)^2]\, c^{\tau'\tau}_{l'l,\, m} - \alpha^{l'}_m \alpha^l_m\, c^{\tau'\tau}_{l'l,\, m-1} - \alpha^{l'}_{m+1}\, \alpha^l_{m+1}\, c^{\tau'\tau}_{l'l,\, m+1}\}\, \xi^{\tau'}_{l',\, m}.$$

The second of the equalities (6) therefore yields a system of equations for determining $c^{\tau'\tau}_{l'l,m}$:

$$2c^{\tau'\tau}_{l'l,\, m} = [(\alpha^l_{m+1})^2 + (\alpha^{l'}_m)]\, c^{\tau'\tau}_{l'l,\, m} - \alpha^{l'}_m\, \alpha^l_m\, c^{\tau'\tau}_{l'l,\, m-1} - \alpha^{l'}_{m+1}\, \alpha^l_{m+1}\, c^{\tau'\tau}_{l'l,\, m+1}.$$

After replacing a^l_m by its various values, we have the system of equations

$$2c^{\tau'\tau}_{l'l,m}=[(l+m+1)(l-m)+(l'+m)(l'-m+1)\,c^{\tau'\tau}_{l'l,m}-$$
$$-\sqrt{(l'+m)(l'-m+1)(l+m)(l-m+1)}]\,c^{\tau'\tau}_{l'l,m-1}-$$
$$-\sqrt{(l'+m+1)(l'-m)(l+m+1)(l-m)}\,c^{\tau'\tau}_{l'l,m+1}. \quad (7)$$

This system of equations can be solved for fixed indices l', l, τ', τ, which then must be given all possible values. Let us fix certain l', l, τ', τ, and write $c^{\tau'\tau}_{l'l,m}$ for the time being simply as c_m. We obtain a system of homogeneous equations for c_m, where $-\min(l', l) \leq m \leq \min(l', l)$ * and the number of equations is equal to the number of unknowns. These equations are most conveniently solved by determining the unknowns one at a time. Giving m its greatest possible value $m_0 = \min(l', l)$, we obtain an equation containing two unknowns c_{m_0} and c_{m_0-1}, of which c_{m_0-1} is determined from c_{m_0}. Now giving m the value m_0-1, we obtain an equation connecting c_{m_0-2}, c_{m_0-1}, c_{m_0}, from which we can again determine c_{m_0-2} in terms of c_{m_0}. Continuing this process, we arrive at the smallest possible value of m, where the equation again will contain only two unknowns, one with the minimum value of m and one with this value plus 1. Since both of these unknowns are already determined by the preceding equations, this relation will either be a consequence of the preceding relations, or it will follow that c_{m_0}, and hence all the unknowns, are equal to zero. These computations show that $c^{\tau'\tau}_{l'l,m}$ can be different from zero only when $|l'-l| \leq 1$, that is, for $l'=l$, $l'=l-1$, and $l'=l+1$. We find $c^{\tau'\tau}_{l'l,m}$ for these cases by the method described above. The proof that $c^{\tau'\tau}_{l'l,m}$ is equal to zero for all other values of l' is very similar, and we leave it to the reader.

We first take $l'=l$, and τ' and τ arbitrary. Then the equations (7) assume the form

$$[2-(l+m+1)(l-m)-(l+m)(l-m+1)]\,c^{\tau'\tau}_{ll,m}+$$
$$+(l+m)(l-m+1)\,c^{\tau'\tau}_{ll,m-1}+(l+m+1)(l-m)\,c^{\tau'\tau}_{ll,m+1}=0.$$

Setting $m=l$, we find that $(1-l)c^{\tau'\tau}_{ll,l}+lc^{\tau'\tau}_{ll,l-1}=0$, hence $c^{\tau'\tau}_{ll,l}=c^{\tau'\tau}_{ll}\cdot l$ and $c^{\tau'\tau}_{ll,l-1}=c^{\tau'\tau}_{ll}(l-1)$, where $c^{\tau'\tau}_{ll}$ is an arbitrary constant, not depending upon m. Setting $m=l-1$, we find similarly that $c^{\tau'\tau}_{ll,l-2}=c^{\tau'\tau}_{ll}(l-2)$. It is easy to see by substitution that the principle here found is quite general, that is, for all m,

$$c^{\tau'\tau}_{ll,m}=c^{\tau'\tau}_{ll}\cdot m.$$

We now set $l'=l-1$. the equations (7) assume the form

$$[2-(l+m+1)(l-m)-(l+m-1)(l-m)]\,c^{\tau'\tau}_{l-1,l,m}+$$
$$+\sqrt{(l+m-1)(l-m)(l+m)(l-m+1)}\,c^{\tau'\tau}_{l-1,l,m-1}+$$
$$+\sqrt{(l+m)(l-m-1)(l+m+1)(l-m)}\,c^{\tau'\tau}_{l-1,l,m+1}=0.$$

In these equations, we make the substitution

$$c^{\tau'\tau}_{l-1,l,m}=\tilde{c}^{\tau'\tau}_{l-1,l,m}\sqrt{(l+m)(l-m)}.$$

After this substitution, and cancelling $\sqrt{(l+m)(l-m)}$ from the m-th equation, we obtain the system

* Since $-l'\leq m'\leq l'$, $-l\leq m\leq l$, and since $c^{\tau'\tau}_{l'l,m'm}\neq 0$ only for $m'=m$, it is clear that m varies from $-\min(l', l)$ to $\min(l', l)$. We note, by the way, that when m assumes its greatest admissible value m_0, one may use the equation formally, since the coefficient of $c^{\tau'\tau}_{l'l,m_0+1}$ is equal to zero.

$$2[1-l^2+m^2]\tilde{c}^{\tau'\tau}_{l-1,l,m}+[l^2-(m-1)^2]\tilde{c}^{\tau'\tau}_{l-1,l,m-1}+ \\ +[l^2-(m+1)^2]\tilde{c}^{\tau'\tau}_{l-1,l,m+1}=0.$$

It is easy to see that this system of equations is satisfied for $\tilde{c}^{\tau'\tau}_{l-1,l,m}$ which is constant and does not depend upon m. Let us set $\tilde{c}^{\tau'\tau}_{l-1,l,m}=c^{\tau'\tau}_{l-1,l}$. From this we find, upon turning to the original unknowns

$$c^{\tau'\tau}_{l-1,l,m}=c^{\tau'\tau}_{l-1,l}\sqrt{l^2-m^2}.$$

Finally, let us set $l'=l+1$. In the equations obtained in this case,

$$[2-(l+m+1)(l-m)-(l+m+1)(l-m+2)]c^{\tau'\tau}_{l+1,l,m}+ \\ +\sqrt{(l+m+1)(l-m+2)(l+m)(l-m+1)}\,c^{\tau'\tau}_{l+1,l,m-1}+ \\ +\sqrt{(l+m+2)(l-m+1)(l+m+1)(l-m)}\,c^{\tau'\tau}_{l+1,l,m+1}=0$$

we make the substitution

$$c^{\tau'\tau}_{l+1,l,m}=\sqrt{(l+m+1)(l-m+1)}\,\tilde{c}^{\tau'\tau}_{l+1,l,m}.$$

After this substitution and cancelling of $\sqrt{(l+m+1)(l-m+1)}$, we come to the equation

$$2[m^2-l^2-2l]\tilde{c}^{\tau'\tau}_{l+1,l,m}+[l^2-m^2+2l+2m]\tilde{c}^{\tau'\tau}_{l+1,l,m-1}+ \\ +[l^2-m^2+2l-2m]\tilde{c}^{\tau'\tau}_{l+1,l,m+1}=0.$$

The solution of this equation is also a constant, independent of m. We denote it by $c^{\tau'\tau}_{l+1,l}$.

In this way, we have found the following values of the elements $c^{\tau'\tau}_{l'l,m}$ of the matrix L_3:

$$\begin{aligned} c^{\tau'\tau}_{l-1,l,m}&=c^{\tau'\tau}_{l-1,l}\sqrt{l^2-m^2}, \\ c^{\tau'\tau}_{ll,m}&=c^{\tau'\tau}_{ll}m, \\ c^{\tau'\tau}_{l+1,l,m}&=c^{\tau'\tau}_{l+1,l}\sqrt{(l+1)^2-m^2}. \end{aligned} \tag{8}$$

We now determine the matrices L_1 and L_2. We denote the elements of the matrix L_1 by $a^{\tau'\tau}_{l'l,m'm}$, that is, we set

$$L_1\xi^{\tau}_{lm}=\sum_{l',m',\tau'}a^{\tau'\tau}_{l'l,m'm}\xi^{\tau'}_{l'm'}. \tag{9}$$

In order to find $a^{\tau'\tau}_{l'l,m'm}$, we use the fact that $L_1=[A_2,L_3]$, and substitute known matrices for A_2 and L_3. We then obtain

$$=A_2\sum_{l',m',\tau'}c^{\tau'\tau}_{l'l,m'm}\xi^{\tau'}_{l'm'}-\frac{1}{2}L_3(\alpha^l_m\xi^{\tau}_{l,m-1}-\alpha^l_{m+1}\xi^{\tau}_{l,m+1})=$$

$$=\frac{1}{2}\sum_{l',m',\tau'}c^{\tau'\tau}_{l'l,m'm}(\alpha^{l'}_{m'}\xi^{\tau'}_{l',m'-1}-\alpha^{l'}_{m'+1}\xi^{\tau'}_{l',m'+1})-\frac{1}{2}\alpha^l_m\sum_{l',m',\tau'}c^{\tau'\tau}_{l'l,m',m-1}\xi^{\tau'}_{l'm'}+ \\ +\frac{1}{2}\alpha^l_{m+1}\sum_{l',m',\tau'}c^{\tau'\tau}_{l'l,m',m+1}\xi^{\tau'}_{l'm'}.$$

Breaking the first sum into two and correspondingly changing the index of summation in each of the sums so obtained, we can write our result in the following way:

$$L_1\xi^{\tau}_{lm}=\frac{1}{2}\sum_{l',\,m',\,\tau'}(\alpha^{l'}_{m'+1}\,c^{\tau'\tau}_{l'l,\,m'+1,\,m}-\alpha^{l'}_{m'}\,c^{\tau'\tau}_{l'l,\,m'-1,m}-$$

$$-\alpha^{l}_{m}\,c^{\tau'\tau}_{l'l,\,m',\,m-1}+\alpha^{l}_{m+1}\,c^{\tau'\tau}_{l'l,\,m',\,m+1})\,\xi^{\tau'}_{l'm'}.$$

Consequently, the elements of the matrix L_1 have the form

$$a^{\tau'\tau}_{l'l,\,m'm}=\frac{1}{2}(\alpha^{l'}_{m'+1}\;c^{\tau'\tau}_{l'l,\,m'+1,\,m}-\alpha^{l'}_{m'}c^{\tau'\tau}_{l'l,\,m'-1,\,m}-\alpha^{l}_{m}\,c^{\tau'\tau}_{l'l,\,m',\,m-1}+$$

$$+\alpha^{l}_{m+1}\,c^{\tau'\tau}_{l'l,\,m',\,m+1}). \tag{10}$$

Since $c^{\tau'\tau}_{l'l,m'm}\neq 0$ only for $m'=m$ and $l'=l-1,\ l,\ l+1$, we find that for fixed m, l, τ', and τ, there are 6 numbers $a^{\tau'\tau}_{l'l,m'm}$, different from 0. These are $a^{\tau'\tau}_{l-1,l,m-1,m}$, $a^{\tau'\tau}_{ll,m-1,m}$, $a^{\tau'\tau}_{l+1,l,m-1,m}$, $a^{\tau'\tau}_{l-1,l,m+1,m}$, $a^{\tau'\tau}_{ll,m+1,m}$, and $a^{\tau'\tau}_{l+1,l,m+1,m}$. Substituting $a^{l}_{m}=\sqrt{(l+m)(l-m+1)}$ and $c^{\tau'\tau}_{l'lm}$ from formula (8) in formula (10), we find the following values for these coefficients:

$$\begin{aligned}
a^{\tau'\tau}_{l-1,\,l,\,m-1,\,m}&=-\frac{c_{l-1,\,l}}{2}\sqrt{(l+m)(l+m-1)},\\
a^{\tau'\tau}_{l,\,l,\,m-1,\,m}&=\frac{c_{ll}}{2}\sqrt{(l+m)(l-m+1)},\\
a^{\tau'\tau}_{l+1,\,l,\,m-1,\,m}&=\frac{c_{l+1,\,l}}{2}\sqrt{(l-m+1)(l-m+2)},\\
a^{\tau'\tau}_{l-1,\,l,\,m+1,\,m}&=\frac{c_{l-1,\,l}}{2}\sqrt{(l-m)(l-m-1)},\\
a^{\tau'\tau}_{l,\,l,\,m+1,\,m}&=\frac{c_{ll}}{2}\sqrt{(l+m+1)(l-m)},\\
a^{\tau'\tau}_{l+1,\,l,\,m+1,\,m}&=-\frac{c_{l+1,\,l}}{2}\sqrt{(l+m+1)(l+m+2)}.
\end{aligned} \tag{11}$$

The constants $c_{l-1,l}$, c_{ll}, and $c_{l+1,l}$ are the same here as in formula (8). In the same way, we can find the matrix $L_2=-[A_1,L_3]$. Setting

$$L_2\xi^{\tau}_{lm}=\sum_{l',\,m',\,\tau'}b^{\tau'\tau}_{l'l,\,m'm}\,\xi^{\tau'}_{l'm'},$$

we find that

$$\begin{aligned}
b^{\tau'\tau}_{l-1,\,l,\,m-1,\,m}&=-\frac{ic_{l-1,\,l}}{2}\sqrt{(l+m)(l+m-1)},\\
b^{\tau'\tau}_{l,\,l,\,m-1,\,m}&=\frac{ic_{ll}}{2}\sqrt{(l+m)(l-m+1)},\\
b^{\tau'\tau}_{l+1,\,l,\,m-1,\,m}&=\frac{ic_{l+1,\,l}}{2}\sqrt{(l-m+1)(l-m+2)},\\
b^{\tau'\tau}_{l-1,\,l,\,m+1,\,m}&=-\frac{ic_{l-1,\,l}}{2}\sqrt{(l-m)(l-m-1)},\\
b^{\tau'\tau}_{l,\,l,\,m+1,\,m}&=-\frac{ic_{ll}}{2}\sqrt{(l+m+1)(l-m)},\\
b^{\tau'\tau}_{l+1,\,l,\,m+1,\,m}&=\frac{ic_{l+1,\,l}}{2}\sqrt{(l+m+1)(l+m+2)}.
\end{aligned} \tag{12}$$

All of the remaining $b^{\tau'\tau}_{l'l,m'm}$ are equal to 0.

In this way, we have found the possible forms for the matrices L_1, L_2, L_3 for systems of invariant equations of the first order. The constants $c^{\tau'\tau}_{l-1,l}$, $c^{\tau'\tau}_{ll}$, and $c^{\tau'\tau}_{l+1,l}$ which appear in these matrices can be chosen arbitrarily. Giving these constants various fixed values, we obtain different invariant systems of equations.

4. Solution of invariant equations. We shall now demonstrate that it is convenient to seek solutions of systems of invariant equations in the form of series of generalized spherical functions. Using the expansions described in the preceding section, we reduce the problem of solving an arbitrary invariant system (1) to the problem of solving a system of ordinary differential equations, much as was done in §8 with Maxwell's equations.

For this purpose, we transform equation (1) in the following way. First, we go over to spherical co-ordinates. At every point, we have

$$\begin{aligned}\frac{\partial}{\partial x_1} &= -\frac{\sin\varphi}{r\sin\vartheta}\frac{\partial}{\partial\varphi} + \frac{\cos\varphi\cos\vartheta}{r}\frac{\partial}{\partial\vartheta} + \cos\varphi\sin\vartheta\frac{\partial}{\partial r},\\ \frac{\partial}{\partial x_2} &= \frac{\cos\varphi}{r\sin\vartheta}\frac{\partial}{\partial\varphi} + \frac{\sin\varphi\cos\vartheta}{r}\frac{\partial}{\partial\vartheta} + \sin\varphi\sin\vartheta\frac{\partial}{\partial r},\\ \frac{\partial}{\partial x_3} &= -\frac{\sin\vartheta}{r}\frac{\partial}{\partial\vartheta} + \cos\vartheta\frac{\partial}{\partial r}.\end{aligned}$$

It follows from the results of §8, and can also be verified immediately, that this change of independent variables changes the system (1) in the same way as the following transformation. At the point with spherical co-ordinates (ϕ, ϑ), we subject space to the rotation $g = g(\frac{\pi}{2} - \phi, \vartheta, \pi)$ (this is the inverse to $g(0, \vartheta, \frac{\pi}{2} + \phi)$) and set

$$\frac{\partial}{\partial x_1'} = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}, \quad \frac{\partial}{\partial x_2'} = -\frac{1}{r}\frac{\partial}{\partial\vartheta}, \quad \frac{\partial}{\partial x_3'} = \frac{\partial}{\partial r}.$$

At the same time, we subject the unknown functions $\psi_1, \psi_2, \cdots, \psi_N$ to the corresponding transformation, setting $\psi' = T_g\psi$, where $T_g = T(\frac{\pi}{2} - \phi, \vartheta, \pi)$. In the system (1), we substitute $\psi = T_{g^{-1}}\psi'$ and then apply the transformation T_g. This produces the system

$$\begin{aligned}T_gL_1\left[-\frac{\sin\varphi}{r\sin\vartheta}\frac{\partial(T_g^{-1}\psi')}{\partial\varphi} + \frac{\cos\varphi\cos\vartheta}{r}\frac{\partial(T_g^{-1}\psi')}{\partial\vartheta} + \cos\varphi\sin\vartheta\frac{\partial(T_g^{-1}\psi')}{\partial r}\right] +\\ + T_gL_2\left[\frac{\cos\varphi}{r\sin\vartheta}\frac{\partial(T_g^{-1}\psi')}{\partial\varphi} + \frac{\sin\varphi\cos\vartheta}{r}\frac{\partial(T_g^{-1}\psi')}{\partial\vartheta} + \sin\varphi\sin\vartheta\frac{\partial(T_g^{-1}\psi')}{\partial r}\right] +\\ + T_gL_3\left[-\frac{\sin\vartheta}{r}\frac{\partial(T_g^{-1}\psi')}{\partial\vartheta} + \cos\vartheta\frac{\partial(T_g^{-1}\psi')}{\partial r}\right] + \varkappa\psi' = 0.\end{aligned}$$

Since the system is invariant, we have the following equalities (see formula (3) of the present section):

$$\begin{aligned}T_g[-L_1\sin\varphi + L_2\cos\varphi]\,T_g^{-1} &= L_1,\\ T_g[L_1\cos\varphi\cos\vartheta + L_2\sin\varphi\cos\vartheta - L_3\sin\vartheta]\,T_g^{-1} &= L_2,\\ T_g[L_1\cos\varphi\sin\vartheta + L_2\sin\varphi\sin\vartheta + L_3\cos\vartheta]\,T_g^{-1} &= L_3.\end{aligned}$$

With the use of these equalities, the transformed system can be written in the form*

$$\frac{1}{r\sin\vartheta}L_1T_g\frac{\partial(T_g^{-1}\psi')}{\partial\varphi} - \frac{1}{r}L_2T_g\frac{\partial(T_g^{-1}\psi')}{\partial\vartheta} + L_3T_g\frac{\partial(T_g^{-1}\psi')}{\partial r} + \varkappa\psi' = 0. \qquad (13)$$

The matrix T_g^{-1} depends upon ϕ and ϑ. Therefore, in differentiating T_g^{-1} with respect to these variables, we must differentiate both factors. This being done, the equation takes on the form

---

* In order to write the system in this form, it is necessary to multiply each matrix L_k on the right by the matrix $L_g^{-1}T_g$.

$$\frac{1}{r\sin\vartheta}L_1\frac{\partial\psi'}{\partial\varphi}-\frac{1}{r}\frac{\partial\psi'}{\partial\vartheta}+L_3\frac{\partial\psi'}{\partial\vartheta}+$$

$$+\left[\frac{1}{r\sin\vartheta}L_1T_g\frac{\partial T_g^{-1}}{\partial\varphi}-\frac{1}{r}L_2T_g\frac{\partial T_g^{-1}}{\partial\vartheta}+\varkappa E\right]\psi'=0. \qquad (14)$$

The expressions $T_g\frac{\partial T_g^{-1}}{\partial\phi}$ and $T_g\frac{\partial T_g^{-1}}{\partial\vartheta}$ which enter in the coefficient of ψ' are nothing else than linear combinations of the matrices A_k which correspond under the representation to infinitesimal rotations around the co-ordinate axes. In fact, recalling that $g=g(\frac{\pi}{2}-\phi,\vartheta,\pi)$ and that $g^{-1}=g(0,\vartheta,\phi+\frac{\pi}{2})$, and writing $T_g\,\partial T_g^{-1}/\partial\phi$, for example, in parametric form, we have

$$T_g\frac{\partial T_g^{-1}}{\partial\varphi}=\lim_{\Delta\varphi\to 0}\frac{1}{\Delta\varphi}T\left(\frac{\pi}{2}-\varphi,\ \vartheta,\ \pi\right)\left[T\left(0,\ \vartheta,\ \varphi+\Delta\varphi+\frac{\pi}{2}\right)-\right.$$

$$\left.-T\left(0,\vartheta,\varphi+\frac{\pi}{2}\right)\right]=\lim_{\Delta\varphi\to 0}\frac{1}{\Delta\varphi}\left[T\left(\frac{\pi}{2}-\varphi,\vartheta,\pi\right)T\left(0,\vartheta,\varphi+\Delta\varphi+\frac{\pi}{2}\right)-E\right].$$

The first summand in brackets is a matrix which obviously has limit E as $\Delta\phi\to 0$. Hence the whole square bracket can be represented as a linear combination of matrices A_k, multiplied by $\Delta\phi+$ terms of higher order (see §2). The limit of the entire expression, that is, $T_g\,\partial T_g^{-1}/\partial\phi$, is simply a linear combination of A_1, A_2, and A_3. Here the coefficients appearing in the linear combination depend upon the rotation $g(\frac{\pi}{2}-\phi,\vartheta,\pi)\,g(0,\vartheta,\phi+\Delta\phi+\frac{\pi}{2})$. Hence they will be the same for the product $T_g\,\partial T_g^{-1}/\partial\phi$ and for the product $g\,\partial g^{-1}/\partial g$. We can compute the last product immediately:

$$g=\left\|\begin{matrix}-\sin\varphi & \cos\varphi & 0\\ -\cos\varphi\cos\vartheta & -\sin\varphi\cos\vartheta & \sin\vartheta\\ \cos\varphi\sin\vartheta & \sin\varphi\sin\vartheta & \cos\vartheta\end{matrix}\right\|,$$

$$g^{-1}=\left\|\begin{matrix}-\sin\varphi & -\cos\varphi\cos\vartheta & \cos\varphi\sin\vartheta\\ \cos\varphi & -\sin\varphi\cos\vartheta & \sin\varphi\sin\vartheta\\ 0 & \sin\vartheta & \cos\vartheta\end{matrix}\right\|,$$

$$g\frac{\partial g^{-1}}{\partial\varphi}=\left\|\begin{matrix}0 & -\cos\vartheta & \sin\vartheta\\ \cos\vartheta & 0 & 0\\ -\sin\vartheta & 0 & 0\end{matrix}\right\|=a_2\sin\vartheta+a_3\cos\vartheta,$$

$$g\frac{\partial g^{-1}}{\partial\vartheta}=\left\|\begin{matrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{matrix}\right\|=a_1.$$

Here a_1, a_2, and a_3 are matrices which correspond to infinitesimal rotations in the principal (3-dimensional) representation.

Consequently, for an arbitrary representation $g\to T_g$, we have the same equalities:

$$T_g\frac{\partial T_g^{-1}}{\partial\varphi}=A_2\sin\vartheta+A_3\cos\vartheta,\quad T_g\frac{\partial T_g^{-1}}{\partial\vartheta}=A_1.$$

Substituting these products in the system (14), we transform (14) into the final form

$$\frac{1}{r\sin\vartheta}L_1\frac{\partial\psi}{\partial\varphi}-\frac{1}{r}L_2\frac{\partial\psi}{\partial\vartheta}+L_3\frac{\partial\psi}{\partial r}+\frac{1}{r}(L_1A_2-L_2A_1+\operatorname{ctg}\vartheta L_1A_3)\psi+\varkappa\psi=0. \qquad (15)$$

We have carried out on the quantity ψ at each point the same transformation which, in §8, preceded the expansion of this quantity in generalized spherical functions. It is to be recalled that this expansion is invariant with respect to rotations.

We now expand every component ψ^{τ}_{lm} of the quantity ψ in a series of generalized spherical functions. In view of the invariance of the system of equations, we may suppose that the individual terms of the series will satisfy the system of equations. But this, as we shall show later, leads to separation of the variables and reduces our system to a system of ordinary differential equations. In order to perform the needed computations, we write the system (15) in components. As a preliminary, we compute the matrix $D = L_1A_2 - L_2A_1$.

Setting
$$D\xi^{\tau}_{l,m} = \sum_{l',m',\tau'} d^{\tau'\tau}_{l'l,m'm}\xi^{\tau'}_{l',m'}$$
and using formulas (8), (11), and (12), as well as the expression for A_1 and A_2 from §2, we find that
$$\left.\begin{aligned} d^{\tau'\tau}_{l-1,l,mm} &= c^{\tau'\tau}_{l-1,l}(l-1)\sqrt{l^2-m^2},\\ d^{\tau'\tau}_{ll,mm} &= c^{\tau'\tau}_{ll}m,\\ d^{\tau'\tau}_{l+1,l,mm} &= -c^{\tau'\tau}_{l+1,l}\,l\sqrt{(l+1)^2-m^2}. \end{aligned}\right\} \quad (16)$$

The remaining $d^{\tau'\tau}_{l'l,m'm}$ are equal to zero. The system (15) in the components ψ^{τ}_{lm} has the form
$$\frac{1}{r\sin\varphi}\sum_{l',m',\tau'} a^{\tau\tau'}_{ll',mm'}\frac{\partial\psi^{\tau'}_{l'm'}}{\partial\varphi} - \frac{1}{r}\sum_{l',m',\tau'} b^{\tau\tau'}_{ll',mm'}\frac{\partial\psi^{\tau'}_{l'm'}}{\partial\vartheta} + \sum_{l',m',\tau'} c^{\tau\tau'}_{ll',mm'}\frac{\partial\psi^{\tau'}_{l'm'}}{\partial r} +$$
$$+\frac{1}{r}\sum_{l',m',\tau'} d^{\tau\tau'}_{ll',mm'}\psi^{\tau'}_{l'm'} - \frac{i}{r}\operatorname{ctg}\vartheta\sum_{l',m',\tau'} m' a^{\tau\tau'}_{ll',mm'}\psi^{\tau'}_{l'm'} + \varkappa\psi^{\tau}_{lm} = 0\,.^{*} \quad (17)$$
The coefficients $a^{\tau\tau'}_{ll',mm'}$, $b^{\tau\tau'}_{ll',mm'}$, $c^{\tau\tau'}_{ll',mm'}$, and $d^{\tau\tau'}_{ll',mm'}$ are defined by formulas (8), (11), (12), and (16) of the present section. In the first, second, and fifth sums, there are 6 summands different from zero, and in the third and fourth there are 3.

In order to separate the variables, we now set
$$\psi^{\tau}_{lm} = f^{l_0}_{lm\tau}(r)\,T^{l_0}_{mn}\left(\frac{\pi}{2}-\varphi,\ \vartheta,\ 0\right),$$
where $l_0 \geq l$ and $-l_0 \leq m,\ n \leq l_0$. We then substitute these products in the system (17). It then turns out that $T^{l_0}_{m-1,n}$ and $T^{l_0}_{m+1,n}$, which appear in the first, second, and fifth sums, can be expressed in terms of $T^{l_0}_{mn}$, by means of the recurrence formulas. Then $T^{l_0}_{mn}$ is cancelled from each equation, and there remain in each equation only functions of r.

We next carry out the substitution of the values of the coefficients
$$a^{\tau\tau'}_{ll',mm'},\quad b^{\tau\tau'}_{ll',mm'},\quad c^{\tau\tau'}_{ll',mm'},\quad d^{\tau\tau'}_{ll',mm'},$$

---

[*] One must take into account the fact that when the matrices L_k act on the vectors ξ^{τ}_{lm}, the summation is carried out over the first indices of each pair. Hence, when the functions ψ^{τ}_{lm} are transformed, it it necessary to sum over the second indices.

and collect terms with one and the same function of r.

We obtain in this fashion the system

$$
\begin{aligned}
&\sum_{\tau'} c^{\tau\tau'}_{l,l-1}\left\{\left[\sqrt{(l^2-m^2)}\,\frac{df^{l_0}_{l-1,m,\tau'}}{dr}-\frac{(l-1)\sqrt{(l^2-m^2)}}{r}\,f^{l_0}_{l-1,m,\tau'}\right]T^{l_0}_{m,n}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right)-\right.\\
&-\frac{i}{2r}\sqrt{(l+m-1)(l+m)}\,f^{l_0}_{l-1,m-1\tau'}\left[\frac{1}{\sin\vartheta}\frac{\partial T^{l_0}_{m-1,n}}{\partial\varphi}+\frac{\partial T^{l_0}_{m-1,n}}{\partial\vartheta}-\frac{(m-1)\cos\vartheta}{\sin\vartheta}T^{l_0}_{m-1,n}\right]-\\
&\qquad-\frac{i}{2r}\sqrt{(l-m-1)(l-m)}\,f^{l_0}_{l-1,m+1,\tau'}\left[-\frac{1}{\sin\vartheta}\frac{\partial T^{l_0}_{m+1,n}}{\partial\varphi}+\frac{\partial T^{l_0}_{m+1,n}}{\partial\vartheta}+\right.\\
&\qquad\left.\left.+\frac{(m+1)\cos\vartheta}{\sin\vartheta}T^{l_0}_{m+1,n}\right]\right\}+c^{\tau\tau'}_{ll}\left\{\left[m\,\frac{df^{l_0}_{lm\tau'}}{dr}+\frac{m}{r}f^{l_0}_{lm\tau}\right]T^{l_0}_{mn}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right)+\right.\\
&\qquad+\frac{i}{2r}\sqrt{(l+m)(l-m+1)}\,f^{l_0}_{l,m-1,\tau'}\left[\frac{1}{\sin\vartheta}\frac{\partial T^{l_0}_{m-1,n}}{\partial\varphi}+\frac{\partial T^{l_0}_{m-1,n}}{\partial\vartheta}-\right.\\
&-\left.\frac{(m-1)\cos\vartheta}{\sin\vartheta}T^{l_0}_{m-1,n}\right]-\frac{i}{2r}\sqrt{(l+m+1)(l-m)}\,f^{l_0}_{l,m+1,\tau'}\left[-\frac{1}{\sin\vartheta}\frac{\partial T_{m-1,n}}{\partial\varphi}+\right.\\
&\quad\left.\left.+\frac{\partial T_{m-1,n}}{\partial\vartheta}+\frac{(m+1)\cos\vartheta}{\sin\vartheta}T^{l_0}_{m+1,n}\right]\right\}+c^{\tau\tau'}_{l,l+1}\left\{\left[\sqrt{(l+1)^2-m^2}\,\frac{df^{l_0}_{l+1,m,\tau'}}{dr}+\right.\right.\\
&+\left.\frac{l\sqrt{(l+1)^2-m^2}}{r}f^{l_0}_{l+1,m,\tau'}\right]T^{l_0}_{m,n}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right)+\frac{i}{2r}\sqrt{(l-m+2)(l-m+1)}\times\\
&\qquad\times f^{l_0}_{l+1,m-1,\tau'}\left[\frac{1}{\sin\vartheta}\frac{\partial T_{m-1,n}}{\partial\varphi}+\frac{\partial T_{m-1,n}}{\partial\vartheta}-\frac{(m-1)\cos\vartheta}{\sin\vartheta}T^{l_0}_{m-1,n}\right]+\\
&\qquad+\frac{i}{2r}\sqrt{(l+m+2)(l+m+1)}\,f^{l_0}_{l+1,m+1,\tau'}\left[-\frac{1}{\sin\vartheta}\frac{\partial T_{m+1,n}}{\partial\varphi}+\frac{\partial T^{l_0}_{m+1,n}}{\partial\vartheta}+\right.\\
&\qquad\qquad\left.\left.+\frac{(m+1)\cos\vartheta}{\sin\vartheta}T^{l_0}_{m+1,n}\right]\right\}+i\varkappa f^{l_0}_{lm\tau}(r)\,T^{l_0}_{mn}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right)=0.
\end{aligned}
$$

Each equation of the system now at hand contains the 3 generalized spherical functions $T^{l_0}_{mn}$, $T^{l_0}_{m-1,n}$, and $T^{l_0}_{m+1,n}$. However, the recurrence formulas of the appendix to §7 can be applied to the expressions in square brackets containing $T^{l_0}_{m+1,n}$ and $T^{l_0}_{m-1,n}$. These expressions in square brackets can also be expressed in terms of $T^{l_0}_{mn}$.

In fact, recalling that $T^{l_0}_{m-1,n}(\frac{\pi}{2}-\phi,\vartheta,0)=e^{-in(\frac{\pi}{2}-\phi)}u^{l_0}_{mn}(\vartheta)$ and that consequently, $\partial T_{m\pm1,n}/\partial\phi=inT_{m\pm1}$, we can rewrite the square brackets in the form $e^{in(\frac{\pi}{2}-\phi)}(du_{m-1,n}/d\vartheta+[(n-(m-1)\cos\vartheta)/\sin\vartheta]u_{m-1,n})$ and analogously $e^{in(\frac{\pi}{2}-\phi)}(du_{m+1,n}/d\vartheta+[(n-(m+1)\cos\vartheta)/\sin\vartheta]u_{m+1,n})$. But in agreement with formulas (2) of the appendis to §7, the first of these brackets must be equal to $-i\sqrt{(l_0+m)(l_0-m+1)}\,u^{l_0}_{mn}$, and the second equal to $-i\sqrt{(l_0+m+1)(l_0-m)}\,u^{l_0}_{mn}$.

Carrying out the indicated change and denoting $e^{in(\frac{\pi}{2}-\phi)}u^{l_0}_{mn}(\vartheta)$ again by $T^{l_0}_{mn}$, we can cancel $T^{l_0}_{mn}(\frac{\pi}{2}-\phi,\vartheta,0)$ from the entire equation. We obtain as a result a system containing only $f^{l_0}_{lm\tau}(r)$, as follows:

$$\begin{gathered}
\sum_{\tau'} c^{\tau\tau'}_{l,\,l-1}\Bigl[\sqrt{(l^2-m^2)}\,\frac{df^{l_0}_{l-1,\,m,\,\tau'}}{dr}-\frac{(l-1)\sqrt{(l^2-m^2)}}{r}\,f^{l_0}_{l-1,\,m,\,\tau'}-\\
-\frac{1}{2r}\sqrt{(l+m-1)(l+m)}\sqrt{(l_0+m)(l_0-m+1)}\,f^{l_0}_{l-1,\,m-1,\,\tau'}-\\
-\frac{1}{2r}\sqrt{(l-m-1)(l-m)}\sqrt{(l_0+m+1)(l_0-m)}\,f^{l_0}_{l-1,\,m+1,\,\tau'}\Bigr]+\\
+c^{\tau\tau'}_{ll}\Bigl[m\,\frac{df^{l_0}_{lm\tau'}}{dr}+mf^{l_0}_{lm\tau'}+\frac{1}{2r}\sqrt{(l+m)(l-m+1)}\sqrt{(l_0+m)(l_0-m+1)}\,\times\\
\times f^{l_0}_{l,\,m-1,\,\tau'}(r)-\frac{1}{2r}\sqrt{(l+m+1)(l-m)}\sqrt{(l_0+m+1)(l_0-m)}\,f^{l_0}_{l,\,m+1,\,\tau'}(r)\Bigr]+\\
+c^{\tau\tau'}_{l,\,l+1}\Bigl[\sqrt{(l+1)^2-m^2}\,\frac{df^{l_0}_{l+1,\,m,\,\tau'}}{dr}+\frac{l\sqrt{(l+1)^2-m^2}}{r}\,f^{l_0}_{l+1,\,m,\,\tau'}(r)+\\
+\frac{1}{2r}\sqrt{(l-m+2)(l-m+1)}\sqrt{(l_0+m)(l_0-m+1)}\,f^{l_0}_{l+1,\,m-1,\,\tau'}(r)+\\
+\frac{1}{2r}\sqrt{(l+m+2)(l+m+1)}\sqrt{(l_0+m+1)(l_0-m)}\,f^{l_0}_{l+1,\,m+1,\,\tau}\Bigr]+\\
+i\varkappa f^{l_0}_{lm\tau}(r)=0. \qquad (18)
\end{gathered}$$

5. Solution of Dirac's equations. Dirac's system of equations are a special case of the system (18). After deletion of a factor depending on time, Dirac's equations can be written in the form

$$\begin{gathered}
\left(\frac{\partial}{\partial x}+i\,\frac{\partial}{\partial y}\right)u_4+\frac{\partial}{\partial z}\,u_3-\frac{i}{hc}\,[E-mc^2+V(r)]\,u_1=0,\\
\left(\frac{\partial}{\partial x}-i\,\frac{\partial}{\partial y}\right)u_3-\frac{\partial}{\partial z}\,u_4-\frac{i}{hc}\,[E-mc^2+V(r)]\,u_2=0,\\
\left(\frac{\partial}{\partial x}+i\,\frac{\partial}{\partial y}\right)u_2+\frac{\partial}{\partial z}\,u_1-\frac{i}{hc}\,[E+mc^2+V(r)]\,u_3=0,\\
\left(\frac{\partial}{\partial x}-i\,\frac{\partial}{\partial y}\right)u_1-\frac{\partial}{\partial z}\,u_2-\frac{i}{hc}\,[E+mc^2+V(r)]\,u_4=0.
\end{gathered}$$

Under rotations, the pair of functions (u_1, u_2) and the pair of functions (u_3, u_4) are transformed according to the representation of the rotation group with $l = 1/2$. An arbitrary invariant system relative to these unknown functions depends upon the choice of four constants $c^{11}_{1/2,1/2}$, $c^{12}_{1/2,1/2}$, $c^{21}_{1/2,1/2}$, and $c^{22}_{1/2,1/2}$. We obtain Dirac's equations by setting $c^{11}_{1/2,1/2} = c^{22}_{1/2,1/2} = 0$; $c^{12}_{1/2,1/2} = (2hc / E - mc^2 + V(r))$; $c^{21}_{1/2,1/2} = (2hc / E + mc^2 + V(r))$, $\varkappa = -l$. The transformations performed in paragraph 4 on a general invariant system of equations lead in the present case to the following system of ordinary differential equations:

$$\left.\begin{aligned}
\frac{df^l_3}{dr}+\frac{1}{r}\,f^l_3(r)-\frac{i\left(l+\frac{1}{2}\right)}{r}\,f^l_4(r)-k_1f^l_1(r)=0,\\
-\frac{df^l_4}{dr}-\frac{1}{r}\,f^l_4(r)-\frac{i\left(l+\frac{1}{2}\right)}{r}\,f^l_3(r)-k_1f^l_2(r)=0,\\
\frac{df^l_1}{dr}+\frac{1}{r}\,f^l_1(r)-\frac{i\left(l+\frac{1}{2}\right)}{r}\,f^l_2(r)-k_2f^l_3(r)=0,\\
-\frac{df^l_2}{dr}-\frac{1}{r}f^l_2(r)-\frac{i\left(l+\frac{1}{2}\right)}{r}f^l_1(r)-k_2f^l_4(r)=0.
\end{aligned}\right\} \qquad (19)$$

The factors k_1 and k_2 are defined by

$$k_1=\frac{i}{hc}[E-mc^2+V(r)],\quad k_2=\frac{i}{hc}[E+mc^2+V(r)].$$

This system with four unknown functions can be changed into a system of two equations of the second order. For this purpose, we note that if we set $f_2^l(r)=\mp i f_1^l(r)$ and $f_4^l(r)=\pm i f_3^l(r)$, then the second equation becomes identical with the first and the third with the fourth. The system then assumes the form

$$\left.\begin{aligned} &\frac{df_3^l}{dr}+\frac{l+\frac{3}{2}}{r}f_3^l(r)-k_1f_1^l(r)=0.\\ &\frac{df_1^l}{dr}-\frac{l-\frac{1}{2}}{r}f_1^l(r)-k_2f_3^l(r)=0.\end{aligned}\right\} \tag{20}$$

Let (f_1^l, f_3^l) be a solution of the system (20). We obtain two linearly independent solutions $(f_1^l, -if_1^l, f_3^l, if_3^l)$ and $(f_1^l, if_1^l, f_3^l, -if_3^l)$ of the original system (19) from this solution of (20).

In particular, if $V(r)=0$, then a solution of the system (20) is expressed in terms of cylinder functions of half-integer order:

$$f_1^l(r)=\frac{c_1}{\sqrt{r}}J_l(\sqrt{-k_1k_2}\cdot r)+\frac{c_2}{\sqrt{r}}J_{-l}(\sqrt{-k_1k_2}\cdot r),$$

$$f_3^l(r)=-\frac{c_1}{k_2\sqrt{r}}J_{l+1}(\sqrt{-k_1k_2}\cdot r)+\frac{c_2}{k_2\sqrt{r}}J_{-l-1}(\sqrt{-k_1k_2}\cdot r).$$

Thus the solutions which we have found for Dirac's equations have the form

$$\begin{aligned} u_1(r,\varphi,\vartheta)&=f_1^l(r)\,T^l_{\frac{1}{2},n}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right),\\ u_2(r,\varphi,\vartheta)&=\mp if_1^l(r)\,T^l_{-\frac{1}{2},n}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right),\\ u_3(r,\varphi,\vartheta)&=f_3^l(r)\,T^l_{\frac{1}{2},n}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right),\\ u_4(r,\varphi,\vartheta)&=\pm if_3^l(r)\,T^l_{-\frac{1}{2},n}\left(\frac{\pi}{2}-\varphi,\vartheta,0\right)\end{aligned}$$

$$\left(l=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots;\quad n=-l,-l+1,\ldots,l\right).$$

Here we have

$$T^l_{\pm\frac{1}{2},n}(\varphi_1\,\vartheta,\varphi_2)=e^{\mp i\frac{\varphi_1}{2}}\cdot P^l_{\pm\frac{1}{2},n}(\cos\vartheta)\cdot e^{-in\varphi_2},$$

$$P^l_{\pm\frac{1}{2},n}(\mu)=A(1-\mu)^{-\frac{2n\mp1}{4}}(1+\mu)^{-\frac{2n\pm1}{4}}\cdot\frac{d^{l-n}}{d\mu^{l-n}}\left[(1-\mu)^{l\mp\frac{1}{2}}(1+\mu)^{l\pm\frac{1}{2}}\right],$$

$$A=\frac{(-1)^{l\mp\frac{1}{2}}i^{n\pm\frac{1}{2}}}{2^l\left(l\mp\frac{1}{2}\right)}\sqrt{\frac{\left(l\mp\frac{1}{2}\right)!\,(l+n)!}{\left(l\pm\frac{1}{2}\right)!\,(l-n)!}}.$$

The general solution of Dirac's equations can be obtained as a series of these particular solutions.

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